Dynamics of Alfvén Waves in Tokamaks

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Summary. — Theoretical analyses predict that shear Alfvén waves may be driven unstable by energetic particles (with energies in the MeV range) in thermonuclear plasmas, e.g., by fusion products. Indeed, these waves have been experimentally observed and — in certain circumstances — found to be responsible of significant energetic particle losses. This fact, together with the possible detrimental effect of these instabilities on the plasma performance — in the perspective of a fusion reactor —, has attracted significant attention on the topic. The present review article is focused on both linear stability and non-linear dynamics of shear Alfvén in tokamaks, the presently most successful experimental machines devoted to the study of fusion reactions via magnetic confinement of thermonuclear plasmas. The present theoretical investigation highlights both analytical and numerical approaches to the problem.

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1. – Introduction

The international research program on controlled thermonuclear fusion and its efforts to achieve reactor relevant conditions in magnetically confined plasmas have reached their best results, to date, in experiments on machines of the tokamak type. These are experimental devices characterized by a toroidal symmetry, which is invariant for arbitrary rotations in the azimuthal angle about a symmetry axis, often called the toroidal axis. In a tokamak, the plasma is confined by a strong azimuthal (toroidal) magnetic field of the order of some Tesla. Plasma equilibrium, furthermore, requires also the presence of a poloidal magnetic field, which is much smaller than the toroidal field and is provided by a current flowing, in the plasma itself, along the azimuthal direction.
Thermionuclear fusion is expected to occur in a tokamak plasma via deuterium-tritium (DT) reactions

\[ D + T \rightarrow ^4\text{He}(3.52 \text{ MeV}) + n(14.06 \text{ MeV}) , \]

which produce alpha particles and neutrons. The optimal operation conditions for a fusion reactor are those in which fusion alpha particles are confined in the plasma and provide the required energy input to keep the plasma in steady state. This plasma condition is called ignition and is characterized by a self-sustained burning plasma which provides energy via the escaping thermionuclear neutrons. At present, no experimental device has been capable to reach ignition, although the largest existing tokamaks have approached, and essentially reached, the so called breakeven condition, in which the total fusion power (from thermionuclear neutrons and alphas) balances the required power input to maintain the plasma in steady state. The study and characterization of ignited plasmas is the primary goal of an international collaboration project: the International Tokamak Experimental Reactor (ITER) [1].

Fusion power production has been experimentally demonstrated, to date, in two of the largest existing tokamaks, the Joint European Torus [2] (JET) and the Tokamak Fusion Test Reactor [3] (TFTR), of which only JET is still in operation. JET has produced, for the first time, 1.7 MW of DT fusion power in the Preliminary Tritium Experiment [4] (PTE, 1992). After JET PTE, other experiments have followed on TFTR, using optimal DT fuel mixtures and producing up to 10.7 MW of fusion power [5]. More recently, after TFTR decommissioning, JET has obtained, during its second DT experimental campaign, about 5 MW of DT fusion power for an extended operation of roughly 4 s and a record peak fusion power of 16 MW [6].

As it may be clearly recognized from the considerations made so far, the confinement properties of energetic particles in fusion plasmas are of major importance in determining the performance of current and future generations of machines operating near or at reactor relevant regimes. For example, the good confinement of alpha particles produced in the DT fusion reactions is necessary to ignite the DT plasma. Similarly, energetic ions, produced by radio-frequency waves or by injection of neutral particle beams (with energies \( \approx 100 \text{ keV} \)), must be well confined in order to successfully achieve plasma heating and/or current drive. However, while the theoretical predictions of the energetic-particle losses due to Coulomb collisions are sufficiently adequate, there remain serious concerns over "anomalou" losses induced by collective oscillations spontaneously excited by the energetic particles.

Since the fusion alphas are born at 3.52 MeV mainly in the plasma center, the corresponding pressure profile is peaked. As a consequence, the energetic-particle pressure gradient is a free energy source that can destabilize waves which resonantly interact with the periodic motion of the energetic particles [7, 8, 9]. As typically in the case of pressure-gradient driven modes, the instability growth rates increase with the energetic-particle diamagnetic drift frequency which is proportional to the toroidal mode number \( n \). On the other hand, characteristic frequencies of the energetic-particle motion (e.g., transit and bounce) are estimated to be in the MHz range, similar to that of shear Alfvén waves. This observations thus suggest that high-\( n \) (\( n \gg 1 \)) shear Alfvén waves are the prime candidate for the instabilities.

Shear Alfvén waves in a laboratory plasma are, however, difficult to excite, since energy is needed to bend the magnetic field lines. Moreover, in a sheared magnetic field, the shear Alfvén waves are characterized by a continuous spectrum [10]. Thus, these
waves are highly localized around the surface where \( \omega = k_{\parallel} v_A \) (\( \omega \) is the mode frequency, \( k_{\parallel} \) is the wave vector parallel to the magnetic field and \( v_A = B/\sqrt{4\pi \rho} \) the Alfvén speed, \( \rho \) being the plasma mass density), and strongly stabilized because of phase mixing.

This situation, strictly valid in a slab, is qualitatively modified in toroidal confinement devices due to the poloidal symmetry breaking associated with toroidal magnetic field inhomogeneities over a magnetic surface. The resultant couplings between neighbouring poloidal harmonics produces not only frequency gaps [11] in the continuous shear Alfvén spectrum, but also discrete Alfvén eigenmodes.

These discrete modes, known as Toroidal Alfvén Eigenmodes [12, 13] (TAE's), are localized in the forbidden frequency window (‘gap’) of the shear Alfvén continuum. As a consequence, TAE’s are undamped, to the lowest order, due to their negligible coupling to the continuum.

That TAE's are marginally stable naturally suggests that energetic particles can resonantly destabilize these modes. Furthermore, these resonant energetic particles could also be effectively scattered by the resultant Alfvénic fluctuations. Indeed, it has been shown that even low-amplitude TAE's, with \( \delta B/B \approx 5 \times 10^{-4} \), can cause severe fusion alpha particles losses [14]. The study of the linear stability of TAE’s is, therefore, an important issue for tokamak fusion research, and has attracted increasing theoretical as well as experimental interest [15, 16, 17, 18, 19].

The linear TAE drive, due to the resonant interaction of the mode with the periodic transit of the passing energetic particles has been extensively studied [20, 21, 22, 23]. The weakening effect on the linear drive because of resonance detuning due to finite particle drift orbits has also been considered [24, 25, 26]. Furthermore, it has been pointed out [25, 26, 27] that both passing and trapped (between magnetic mirror points) particles play important roles in determining the linear drive.

A number of damping mechanisms have been suggested by various authors to balance the energetic-particle linear drive and, hence, to determine the marginal stability threshold for TAE's. Electron Landau damping [20, 21, 22, 23] is typically negligible, while ion Landau damping [23] and trapped electron collisional damping [28, 29] could be important depending on the plasma parameters [25]. Also non-ideal effects of electron inertia and finite ion Larmor radius may significantly enhance the damping rate [26, 30, 31, 32, 33, 34, 35] and even yield a new kinetic branch of the TAE modes, the Kinetic Toroidal Alfvén Eigenmodes [30] (KTAE’s). Another effective damping mechanism, due to the coupling of the TAE mode to the continuous shear Alfvén spectrum, was suggested first in ref. [20] and studied in detail in refs. [36, 37, 38, 39] for the high-\( n \) case, and in ref. [40] for low-\( n \). In this case, the damping is a consequence of the toroidal mode coupling, which renders the TAE global radial mode width much broader than the typical radial extent of a single poloidal harmonic [37].

In addition to ideal (TAE) and kinetic (KTAE) discrete plasma eigenmodes of the shear Alfvén spectrum, tokamak plasma may be characterized by forced oscillations in the presence of a sufficiently strong energetic particle free energy source. These forced oscillations, called Energetic Particle Modes [41] (EPM's), are excited via resonant interactions with fast, “hot”, ions [41, 42] and, thus, have the typical frequencies of particle motions, i.e., \( \omega_{1H} \) (the “transit” frequency of fast ions around the toroidal plasma column), \( \omega_{bH} \) (the “bounce” frequency associated with their periodic motion between two magnetic mirror points for “trapped” energetic particles), and/or \( \omega_{ph} \) (the precession rate in the toroidal direction of the trapped particle orbits). Since these forced oscillations have nothing to do with the presence of a frequency gap in the shear Alfvén continuous spectrum, it is readily demonstrated that the threshold in the energetic par-
ticle free energy in order to excite EPM’s is associated with the necessity to balance (at least) the continuum damping due to coupling to the Alfvén continuum [41].

Considering that small fluctuation levels of shear Alfvén oscillations in a tokamak plasma can cause severe fusion alpha particles losses [14], it is not only important to analyze the global stability properties of those waves, but also to understand their non-linear dynamics and the fundamental physical processes which, eventually, yield to saturation of these modes. Essentially, it is possible to classify the saturation mechanisms of Alfvén modes in tokamaks in two major categories: those related to energetic particle non-linearities, i.e., to the non-linear modifications of particle motions; and those associated with mode-mode couplings. The former of these processes are the most studied in the literature, since the first work on non-linear dynamics of shear Alfvén waves in tokamaks [43], and are expected to be the most important for weakly unstable modes. However, it has been recently pointed out that also the latter processes may be important in determining non-linear mode saturation, depending on the considered plasma equilibrium (see, e.g., ref. [44]). In any event, it is difficult to identify, in general, a single specific physical process – falling in one of the two categories mentioned above – that may be thought to be the most relevant in determining the non-linear evolution of shear Alfvén waves in tokamaks and their effect on the energetic particle population. In fact, there may exist other non-linear dynamics, even stronger than the effects discussed above, which may determine the saturated level of these instabilities. This is, e.g., the typical case of EPM’s, as it may have been expected from the very definition of these modes as forced oscillations.

Both linear stability properties of shear Alfvén waves in tokamaks and their non-linear dynamic evolution will be analyzed in the present review article, along with the consequences that these modes may have on energetic particle confinement. Our goal is to give an as complete as possible overview of the major results, obtained in the literature, on these topics. However, our treatment has, by no means, the aim of being complete and exhaustive. We will rather try to summarize the present understanding of the matter, frequently referring to published papers for detailed derivations, and sometimes only quoting results which would require a too specific discussion for the more general viewpoint we have tried to maintain throughout the present work.

2. – Linear Theory

We wish to start the discussion of the general properties of the Alfvén-wave spectrum using simple-model equations, the so-called ideal magnetohydrodynamic (MHD) equations, which in Gaussian units read [45]:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \\
\frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \frac{1}{c} \mathbf{J} \times \mathbf{B}, \\
\frac{d}{dt} \left( \frac{P}{\rho^2} \right) = 0, \\
\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = 0, \\
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\]

(1) \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \)

(2) \( \frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \frac{1}{c} \mathbf{J} \times \mathbf{B}, \)

(3) \( \frac{d}{dt} \left( \frac{P}{\rho^2} \right) = 0, \)

(4) \( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = 0, \)

(5) \( \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \)
\[ \mathbf{v} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \]
\[ \nabla \cdot \mathbf{B} = 0. \]

In the above equations \( \mathbf{v} \) is the fluid velocity, \( \mathbf{J} \) is the plasma current, \( \mathbf{B} \) is the magnetic field, \( \varrho \) is the mass density, \( P \) is the scalar pressure of the plasma, \( \Gamma \) is the ratio of the specific heats, \( c \) is the speed of light, and

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \]

is the convective derivative.

The ideal MHD equations describe the plasma as a single fluid. In particular, eq. (1) describes the time evolution of mass (conservation of the total number of particles). Equation (2) describes the time evolution of momentum, showing that the fluid is subject to inertial, pressure-gradient and magnetic forces. Equation (3) is the equation of state and generally describes the polytropic evolution of the plasma. It may be combined with the continuity equation and written as

\[ \frac{dP}{dt} = -\Gamma P \nabla \cdot \mathbf{v}. \]

Equation (4), the so-called ideal Ohm’s law, describes the plasma as a perfectly conducting fluid (from which the expression “ideal MHD” originates). Note that in the more general case in which plasma resistivity \( \eta \) is considered, eq. (4) is replaced by

\[ \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}. \]

Finally, the last three equations (eqs. (5), (6), (7)) are the low-frequency limit of the Maxwell equations. From eq. (6) it follows

\[ \nabla \cdot \mathbf{J} = 0, \]

which is equivalent to neglect the displacement current in the Maxwell equations. The relation \( \nabla \cdot \mathbf{J} = 0 \) is also called the quasi-neutrality condition, and states that the charge density is locally zero and does not vary in time.

**2. Waves in an infinite homogeneous plasma.** The ideal MHD model, in spite of its simplicity, describes a very rich phenomenology, even in its linearized limit. Thus, it would be instructive to consider a very simple configuration to start with.

In this section we consider a homogeneous plasma of infinite extent, in which the equilibrium magnetic field is uniform and directed along the \( \mathbf{e}_z \) axis. No gradients are present at the equilibrium, so that the configuration is described by the following set of equations:

\[ \varrho = \varrho_0, \]
\[ \mathbf{v} = 0, \]
\[ P = P_0, \]
\[ \mathbf{B} = \mathbf{B}_0 = B_0 \mathbf{e}_z, \]
\[ \mathbf{J} = 0. \]
where \( \rho_0, P_0, B_0 \) are constants. Next, we linearize the ideal MHD set of equations around the above described equilibrium, assuming the following form for the generic perturbed quantity \( f \):

\[
 f(r, t) = f_0 e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})}.
\]

For simplicity, we also assume that the wave vector \( \mathbf{k} \) lies in the \((y, z)\) plane; moreover, we indicate with \( k_\perp, k_\parallel \), respectively the perpendicular and parallel component of the wave vector with respect to the equilibrium magnetic field. From the linearized MHD equations, solving for the three components \((\delta v_x, \delta v_y, \delta v_z)\) of the perturbed velocity \(\delta \mathbf{v}\), it follows that

\[
 (\omega^2 - k_\parallel^2 v_A^2) \delta v_x = 0,
\]

\[
 (\omega^2 - k_\perp v_S^2) \delta v_y - k_\perp k_\parallel v_S^2 \delta v_z = 0,
\]

\[
 (\omega^2 - k_\parallel^2 v_A^2) \delta v_y + k_\perp k_\parallel v_S^2 \delta v_z = 0,
\]

where \( k^2 \equiv k_\perp^2 + k_\parallel^2 \), \( v_A \equiv (B_0^2/(4\pi \rho_0))^{1/2} \) is the Alfvén velocity and \( v_S \equiv (\Gamma P_0/\rho_0)^{1/2} \) is the sound speed. The dispersion relation is obtained by setting the determinant of the coefficients of the linear system, eqs. (14), to zero:

\[
 \omega^2 = k_\parallel^2 v_A^2, \quad \omega^2 = \frac{1}{2} k^2 (v_A^2 + v_S^2) \left( 1 \pm \sqrt{1 - \frac{\pi^2}{\alpha^2}} \right),
\]

where

\[
 \alpha^2 \equiv 4 \frac{k_\parallel^2}{k^2} \frac{v_A^2 v_S^2}{(v_A^2 + v_S^2)^2}.
\]

Since \( 0 \leq \alpha^2 \leq 1 \), three solutions are obtained from the dispersion relation eqs. (15), each of them corresponding to a purely oscillating motion.

The first branch of the dispersion relation eqs. (15),

\[
 \omega^2 = \omega_A^2 \equiv k_\parallel^2 v_A^2,
\]

is known as the shear Alfvén wave. It does not depend on \( k_\perp \) and corresponds to a purely transverse wave, having the perturbed magnetic field parallel to the perturbed velocity and perpendicular to the equilibrium magnetic field \( B_0 \mathbf{e}_z \). The wave travels along the equilibrium field lines with a group velocity

\[
 \mathbf{v}_{g.A} = v_A \mathbf{B}_0/B_0.
\]

Thus the fluid element and the magnetic field line oscillate in phase (plasma frozen to the magnetic field lines), behaving as a massive elastic string under tension. From the Ohm’s law, eq. (4), it follows that the fluid motion is generated by the \( \mathbf{v}_E \equiv (c/B^2) \mathbf{E} \times \mathbf{B} \) velocity. The motion is such that it makes the magnetic field lines to bend. Moreover, the motion is incompressible \((\nabla \cdot \mathbf{v} = 0)\); thus, the density and pressure perturbations are zero (see eqs. (1) and (9)). The shear Alfvén wave is essentially the result of the balance between the inertia term \((\rho \frac{d \mathbf{v}}{dt})\) and the magnetic field tension \((\mathbf{J} \times \mathbf{B}/c)\) in the momentum equation, eq. (2), producing an oscillation between perpendicular plasma kinetic energy and perpendicular magnetic energy related to the field line bending.

The second and third branches of the dispersion relation are obtained from the second of eqs. (15). These branches correspond to oscillations in which the disturbance depends on both the perpendicular and parallel wave vector components. For both
waves, the motion produces a compression of the magnetic field and a plasma-pressure perturbation \( \nabla \cdot v \neq 0 \).

The second branch, corresponding to the plus sign in front of the square root in eqs. (15), is called the fast magneto-acoustic wave, being characterized by a frequency \( \omega^2 = \omega^2_A \), such that it is always \( \omega^2_A \leq \omega^2_B \). In the limit \( v_B^2/v_A^2 \ll 1 \) – which physically corresponds to the case of plasma \( \beta \equiv 8\pi P_0/B_0^2 \) (the ratio between the plasma kinetic and the magnetic pressures) much less than unity – the fast magneto-acoustic wave is reduced to the so-called compressional Alfvén wave, with

\[
\omega^2_B \approx (k^2_\perp + k^2_\parallel) v_A^2.
\]

In the low-\( \beta \) limit, it can be shown that the ratio between the perturbed plasma pressure and the perturbed magnetic pressure is \( O(\beta) \): thus, the compressional term is dominated by the magnetic field pressure. Furthermore, the fluid is dominated by the perpendicular motion. The fast magneto-acoustic wave essentially results from the balance between plasma inertia and magnetic field tension and compression in the momentum equation, eq. (2). Its group velocity is

\[
v_{gF} = v_A k/k.
\]

The third branch, corresponding to the minus sign in front of the square root in eqs. (15), is called the slow magneto-acoustic wave, for it is characterized by a frequency \( \omega^2 = \omega^2_S \), with \( \omega^2_S \leq \omega^2_A \). Again, in the limit \( \beta \ll 1 \), the slow magneto-acoustic wave is reduced to the so-called sound wave, with

\[
\omega^2_S \approx k^2_\parallel v_S^2.
\]

The fluid is dominated by the parallel motion, and the magnetic field compression is \( O(\beta) \) with respect to that of the fluid plasma. The slow magneto-acoustic wave is essentially the result of the balance between inertia and plasma compression in the momentum equation, eq. (2). Its group velocity is

\[
v_{gS} = v_S B_0/B_0.
\]

It can be shown [45] that the most dangerous modes in confined systems are those corresponding to incompressible motion. In fact, it can be shown that the work done by an arbitrary displacement of the plasma to compress the fluid is always positive (corresponding to an increase of the potential energy), making the system more stable. Hence, the shear Alfvén wave can be considered the most likely one to be driven unstable by the available sources of free energy always present in a confined system. Indeed, if one interprets [46] the terms \( k_\perp v_A \) and \( k_\parallel v_A \) as effective spring constants of the plasma subject to a perturbed motion, in analogy to an harmonic oscillator, one can argue that the shear Alfvén waves are more likely to be driven unstable than the compressional waves, being \( k_\parallel < (k^2_\perp + k^2_\parallel)^{1/2} \). In fact, a larger spring constant can be interpreted as a greater ability of the plasma to maintain its state under an external perturbation. Moreover, it will be shown later that, usually, in the confined systems of interest \( k^2_\parallel \ll k^2_\perp \) (Alfvén waves are typically characterized by parallel wavelength comparable to the system size), thus making the shear Alfvén branch even more dangerous.

Finally, another important difference between shear and compressional Alfvén waves is related to the group velocity. For both branches it is equal to \( v_A \) in magnitude. However, for the compressional wave the group velocity is directed along \( k \) (cf. eq. (20)) and, thus, essentially across the magnetic field (since \( k^2_\perp \gg k^2_\parallel \)), while it is directed
along $B_0$ (cf. eq. (18)) for the shear Alfvén wave, making the latter more suitable to be resonantly excited by fast ions with $v \simeq v_A$. Indeed, at least to the lowest order in the particle gyroradius, the particle motion is directed along $B_0$.

2.2. Waves in a non-uniform slab configuration. – Differently from the case discussed in the previous section, a spatially confined plasma is characterized by equilibrium inhomogeneities. Here we study the simple configuration of a one-dimensional (1-D) plasma slab confined in straight magnetic field [47, 48]. The equilibrium quantities (plasma density, pressure and magnetic field) are assumed to vary only along the $x$ direction \( \rho = \rho(x) = \rho_0(x), \ P = P(x) = P_0(x), \ B = B(x) = B_0(x) \). The equilibrium magnetic field is assumed to have a shear component,

\[
B_0(x) = B_{0y}(x) e_y + B_{0z}(x) e_z.
\]

The equilibrium pressure balance can be found using eqs. (2) and (6)

\[
\frac{d}{dx} \left( P_0 + \frac{B_0^2}{8\pi} \right) = 0.
\]

Again, we proceed by linearizing the ideal MHD eqs. (1-7), introducing, for convenience, the displacement vector $\xi$ of the fluid element, defined as

\[
\delta \mathbf{v} = \frac{\partial \xi}{\partial t}.
\]

The Faraday equation, becomes, after using the Ohm’s law, eq. (4),

\[
\delta \mathbf{B} = \nabla \times (\mathbf{\xi} \times \mathbf{B}_0) = (\mathbf{B}_0 \cdot \nabla) \mathbf{\xi} - \mathbf{B}_0 (\nabla \cdot \mathbf{\xi}) - (\mathbf{\xi} \cdot \nabla) \mathbf{B}_0,
\]

where $\delta \mathbf{B}$ is the linear perturbed magnetic field. Moreover, following ref. [48], the momentum equation can be written as

\[
\rho_0 \frac{\partial^2 \mathbf{\xi}}{\partial t^2} = -\nabla \tilde{P} + \frac{1}{4\pi} (\mathbf{B}_0 \cdot \nabla)^2 \mathbf{\xi} - \frac{1}{4\pi} \mathbf{B}_0 (\mathbf{B}_0 \cdot \nabla) (\nabla \cdot \mathbf{\xi}),
\]

where we have defined

\[
\tilde{P} = \delta P + \frac{\delta \mathbf{B} \cdot \mathbf{B}_0}{4\pi}
\]

as the total (kinetic plus magnetic) perturbed pressure. The perturbed plasma pressure can then be obtained in terms of the displacement vector $\mathbf{\xi}$ from eq. (9)

\[
\delta P = -\xi_x \frac{dP_0}{dx} - \Gamma P_0 (\nabla \cdot \mathbf{\xi}).
\]

In the following, the generic perturbed quantity $f$ is assumed to have the form:

\[
f(r, t) = \tilde{f}(x) e^{-i(\omega t - k_y y - k_z z)}.
\]

Moreover, it is convenient to adopt the set of coordinates based on the parallel and perpendicular directions to the equilibrium magnetic field $\mathbf{B}_0$, defining $\mathbf{e}_\parallel \equiv \mathbf{B}_0 / B_0$ and $\mathbf{e}_\perp \equiv \mathbf{e}_\parallel \times \mathbf{e}_x$. Thus, the momentum equation, eq. (27), yields [48]

\[
D_A \xi_\parallel = \frac{4\pi}{B_0^2} i k_\parallel \tilde{P} + i k_\parallel \left( i k_\parallel \xi_\parallel + i k_\perp \xi_\perp + \frac{d}{dx} \xi_x \right),
\]

\[
D_A \xi_\perp = \frac{4\pi}{B_0^2} i k_\perp \tilde{P},
\]

\[
D_A \xi_x = \frac{4\pi}{B_0^2} \frac{d}{dx} \tilde{P},
\]
where $D_A \equiv (\omega^2/v_A^2) - k_{||}^2$ is the local shear Alfvén propagator. Note that $k_{||} = (k_y B_{0y} + k_z B_{0z})/B_0$ and $k_\perp = (k_y B_{0y} - k_z B_{0z})/B_0$. Using eqs. (24), (26) and (28), one can write $\tilde{p}$ in terms of the three components of $\xi$,

\begin{equation}
\tilde{p} = -i k_{||} \Gamma P_0 \xi_{||} - \left( \Gamma P_0 + \frac{B_0^2}{4\pi} \right) \left( i k_{\perp} \xi_{\perp} + \frac{d}{dx} \xi_x \right).
\end{equation}

Solving eq. (31) for $\xi_{||}$ and substituting the obtained expression into eq. (34), from eq. (32), it is possible to relate $\xi_{\perp}(x)$ to $\xi_x(x)$, as

\begin{equation}
\xi_{\perp}(x) = \frac{\Gamma k_{||}^2}{\omega^2 - D_A} \frac{d \xi_x}{dx},
\end{equation}

where

\begin{equation}
\bar{\omega}(x) \equiv 1 + \frac{\Gamma k_{||}^2}{2 \omega^2 - \Gamma k_{||}^2 v_A^2}.
\end{equation}

Differentiating eq. (32) with respect to $x$, making use of eq. (35) and substituting in eq. (33), we obtain the wave equation for $\xi_x$ [48]:

\begin{equation}
\frac{d}{dx} \left( \frac{B_0^2 D_A \bar{\omega}}{\omega^2 - D_A} \frac{d \xi_x}{dx} \right) - B_0^2 D_A \xi_x = 0.
\end{equation}

Equation (37) describes again the three branches studied in the sect. 2.1, which are now coupled together by equilibrium non-uniformities. It has to be noted that the differential equation, eq. (37), and hence its solution $\xi_x$, is singular at the points where $B_0^2 D_A \bar{\omega} = 0$, which correspond to the appearance of two continuous spectra defined by

\begin{equation}
\omega^2 = \omega_A^2(x) \equiv k_{||}^2(x)v_A^2(x),
\end{equation}

and

\begin{equation}
\omega^2 = \omega_S^2(x) \equiv \frac{v_S^2(x) k_{||}^2(x)}{1 + v_S^2(x)/v_A^2(x)}.
\end{equation}

To further gain some insight on the properties of the continuous spectrum and, in general, on the shear Alfvén waves, we can proceed assuming, as above, that the equilibrium plasma density varies along $x$, $\varrho_0 = \varrho_0(x)$, but taking now a uniform equilibrium magnetic field without shear along the $z$ coordinate $B_0 = B_0 \mathbf{e}_z$, with the plasma being confined by two perfectly conducting plates at $z = 0$ and $z = L$ [49]. Moreover, let us adopt the slow sound wave approximation ($\Gamma \beta \to 0$, or $\bar{\omega} \to 1$ in eq. (37)) to reduce eq. (37) to the description of the coupled shear and compressional Alfvén waves. Note that in this limit it is sufficient to consider the perpendicular components of the momentum equation, eq. (27), which does not depend on the parallel component of the displacement $\xi_{||}$,

\begin{equation}
D_A \xi_{\perp} = \nabla \cdot \left( \frac{\delta B_{||}}{B_0} \right),
\end{equation}

with the parallel component of the perturbed magnetic field given, from eq. (26), by

\begin{equation}
\frac{\delta B_{||}}{B_0} = \nabla \cdot \xi_{\perp}.
\end{equation}
Furthermore, the local shear Alfvén operator $D_A$ is expressed by

$$D_A = \frac{\omega^2}{v_A^2} + \frac{\partial^2}{\partial \xi^2}$$

and $\nabla_{\perp} = e_y k_y + e_x (\partial / \partial x)$. In the remainder of this section, we strictly follow ref. [49].

For convenience, all perturbed fields $f$ are assumed to be decomposed in the complete orthonormal set of shear Alfvén eigenfunctions $\psi_{A\ell}(z|x)$

$$f(r, t) = e^{-i(\omega t - k_y y)} \sum_{\ell=1}^{\infty} \hat{f}_{\ell}(x_0, x_1) \psi_{A\ell}(z|x_1),$$

where $\psi_{A\ell}(z|x)$ satisfies the differential equation

$$\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega_A^2(x)}{v_A^2} \right] \psi_{A\ell}(z|x) = 0,$$

along with boundary conditions $\psi_{A\ell} = 0$ at $z = 0$ and $L$, and $\omega_{A\ell}^2(x)$ defines the local shear Alfvén eigenfrequency. Note that in eq. (43), the existence of continuous frequency spectra, with a corresponding singular eigenfunction behaviour, has been explicitly taken into account by introducing “fast” $x_0$ (singular) and “slow” $x_1$ (equilibrium) radial variables. Using the following definition of the internal product

$$\langle \psi_{A\ell} | \psi_{A\ell'} \rangle \equiv \int_0^\infty v_A^2 \psi_{A\ell} \psi_{A\ell'} dz = \delta_{\ell, \ell'},$$

it is straightforwardly demonstrated that, in the present case,

$$\psi_{A\ell}(z|x) = (2/L v_A^2)^{1/2} \sin k_z z,$$

with $k_z = \ell \pi / L$ and $\omega_{A\ell}^2(x) = k_z^2 v_A^2(x)$.

We now exploit the existence of two spatial scales ($x_0$ and $x_1$) and solve eqs. (40) and (41) via asymptotic expansions of the fluctuating fields $f_{\ell} = \hat{f}_{\ell}^{(0)} + \hat{f}_{\ell}^{(1)} + \ldots$, where $|\hat{f}_{\ell}^{(1)} / \hat{f}_{\ell}^{(0)}| \approx |\partial_{x_0} / \partial x_1|$, etc. With the definition of eq. (43) and considering that $|D_A| \ll k_y^2$ for a typical shear Alfvén wave (i.e., $k_{y1}^2 \ll k_y^2$), eqs. (40) and (41) may be combined and yield, to the lowest order,

$$\begin{bmatrix} \frac{\partial}{\partial x_0} \epsilon_{A\ell} & \frac{\partial}{\partial x_0} - k_y^2 \epsilon_{A\ell} \end{bmatrix} \hat{\xi}^{(0)}_{\ell\ell} = 0,$$

where $\epsilon_{A\ell} = \omega^2 - \omega_{A\ell}^2(x)$. Equation (47) has solutions which become singular at the radial positions $x_{RT}$ where the “local dispersion relation”, $\epsilon_{A\ell} = 0$ ($\omega^2 = \omega_{A\ell}^2(x_{RT})$), is satisfied; i.e., at those positions where the shear Alfvén continuous spectrum is resonantly excited. A natural definition of the “fast” (singular) variable is $x_0 \equiv x - x_{RT}$, since $\epsilon_{A\ell} = \epsilon_{A\ell}'(x_1)x_0$. In this way, it is readily shown that eq. (47) has solutions that may be written in the form

$$\hat{\xi}^{(0)}_{\ell\ell} = \frac{C_{\ell}(x_1)}{\epsilon_{A\ell}'(x_1) x_0} \ln(x_0).$$

Meanwhile, eqs. (40) and (41) can be used to demonstrate that, at the lowest order, $\nabla \cdot \hat{\xi}^{(0)}_{\ell\ell} = 0$, from which we get

$$\hat{\xi}^{(0)}_{\ell\ell} = \frac{i}{k_y} \frac{\partial_x \hat{\xi}^{(0)}_{\ell\ell}}{\partial x} = \frac{i C_{\ell}(x_1)}{k_y \epsilon_{A\ell}'(x_1) x_0} \frac{1}{x_0}.$$
Using eq. (49) along with the $y$ component of eq. (40), it is possible to show that the compressional component of the magnetic field perturbation is given in the form of eq. (43), i.e.,

$$
\delta B_\parallel = e^{-i(w - k_y y)} \sum_{\ell=1}^{\infty} \hat{\delta}_\parallel(\ell) \psi_{A\ell}(z|x_1),
$$

with the important difference that, here, $\hat{\delta}_\parallel(\ell)$ are functions of the “slow” (equilibrium) radial variable $x_1$ only. More specifically, it is readily demonstrated that $\hat{\delta}_\parallel(\ell)$ is directly related to the functions $C_\ell(x_1)$ introduced above:

$$
\frac{\hat{\delta}_\parallel(\ell)}{B_0} = \frac{\varepsilon'_{A\ell}(x_1) x_0 \tilde{\varepsilon}^{(0)}}{i k_y v_A^2(x_1)} \frac{\psi_{A\ell}(z)}{\omega^2} = \frac{C_\ell(x_1)}{k_y^2 v_A^2(x_1)}.
$$

It is important to recognize the fundamental meaning of eqs. (48), (49) and (51): in a non-uniform plasma, shear and compressional Alfvén waves are coupled together and their coupling is the origin of the singular solutions corresponding to the “local” shear Alfvén oscillations of the continuous spectrum $\omega^2 = \omega^2_{A\ell}(x)$. In fact, as discussed in sect. 21, the group velocity of shear Alfvén waves is directed along the magnetic field lines ($z$-direction in the present case), whereas the compressional wave generally carries energy across the field itself. Thus, the latter one “piles up” wave energy at the radial location where the shear Alfvén spectrum is resonantly excited, explaining the origin of “local singular oscillations” [49].

Resonant excitation allows us to introduce the concept of resonant absorption [48] of shear Alfvén waves. In fact, a finite amount of wave energy can be absorbed at the resonant layer, $x_{RE}$. To see this, we start from the fact that the time-averaged energy absorption rate, $d(W)/dt$, is given by the Poynting energy flux into the infinitely narrow layer at $x_{RE}$. Thus [48],

$$
\frac{d(W)}{dt} = \frac{-e L_y}{8\pi} \text{Re} \left\{ \int_0^L dx \int_{-\infty}^{\infty} \nabla \cdot \left[ \left( \frac{1}{c} \xi \times B_0 \right) \times \delta B^* \right] dx \right\},
$$

where $L_y$ is the system extension in the $y$-direction. Using the definitions of eqs. (45) and (46) along with boundary conditions at $z = 0$ and $z = L$, eq. (52) becomes

$$
\frac{d(W)}{dt} = -\frac{B_0 L_y}{8\pi} \text{Re} \left\{ \sum_{\ell=1}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( -\frac{i}{v_A^2} \hat{\xi}^{(0)}_{A\ell} \hat{\delta}_\parallel(\ell) \right) dx \right\}.
$$

The right hand side (r.h.s.) of eq. (53) does not vanish only for the contribution to the integral at the resonant surface $x_{RE}$. Here, using the causality constraint $\text{Im} \omega \to 0^+$, it is possible to demonstrate that $\hat{\xi}^{(0)}_{A\ell}(x_{RE})$ (cf. eq. (48)) has a jump given by

$$
\Delta \hat{\xi}^{(0)}_{A\ell} = -i\pi \text{sgn} \left( \frac{\omega}{\varepsilon'_{A\ell}(x_{RE})} \right) \frac{C_\ell(x_{RE})}{\varepsilon'_{A\ell}(x_{RE})}.
$$

Thus, recalling eq. (51), eq. (53) finally becomes [49]

$$
\frac{d(W)}{dt} = \frac{L_y (\omega A\ell(x_{RE}))}{8\varepsilon'_{A\ell}(x_{RE})} \left( k_y^2 \sum_{\ell=1}^{\infty} \left| \hat{\delta}_\parallel(\ell)(x_{RE}) \right|^2 \right).
$$
The existence of the resonant energy absorption mechanism becomes evident also when one analyzes the time asymptotic response of the system to initial perturbations. From the previous discussion, it can be conjectured that, as $\omega_{A\ell} t \to \infty$, one essentially has $|\partial x| \gg |k_y|$ and, thus, that the relevant equation which describes the time asymptotic response is (cf. eq. (47)):

$$\frac{\partial}{\partial x} \left[ \frac{\partial^2}{\partial x^2} + \omega_{A\ell}^2(x) \right] \frac{\partial}{\partial x} \xi_{x\ell}(x, t) = 0. \tag{55}$$

Equation (55) can be straightforwardly integrated once and it yields

$$\frac{\partial}{\partial x} \xi_{x\ell}(x, t) = \tilde{C}_{\ell}(x) e^{\pm i\omega_{A\ell}(x)t},$$

where $\tilde{C}_{\ell}(x)$ is a function depending on equilibrium non-uniformities. Now, note that, as $\omega_{A\ell} t \to \infty$,

$$\partial_x \cong \pm i\omega_{A\ell}(x) t \tag{56}$$

and, thus,

$$\xi_{x\ell}(x, t) = \mp \frac{\tilde{C}_{\ell}(x)}{\omega_{A\ell}(x)t} e^{\pm i\omega_{A\ell}(x)t}. \tag{57}$$

Meanwhile, noting eq. (49), one readily derives

$$\hat{\xi}_{y\ell}(x, t) = \frac{i}{k_y} \tilde{C}_{\ell}(x) e^{\pm i\omega_{A\ell}(x)t}. \tag{58}$$

Equations (57) and (58) give us further insight in the dynamics associated with the resonant excitation of the shear Alfvén continuum and resonant wave absorption: the $\xi_{x\ell}$ component exhibits the characteristic $(1/t)$ decay via phase mixing of the continuous spectrum, whereas $\hat{\xi}_{y\ell}$ shows undamped oscillations at frequencies corresponding to the shear Alfvén continuum. This peculiar feature will be analyzed in detail in sect. 2.4, where numerical computations are presented, focusing on the resonant excitation of the continuous spectrum. Furthermore, eq. (56) suggests that the radial wave-vector is

$$|k_x| \cong |\omega_{A\ell}(x)t|$$

and, thus, $|k_x| \to \infty$ as $t \to \infty$, in agreement with eq. (48) and with the fact that the wave function becomes singular in the asymptotic time limit.

The wave function singularity that emerges in the asymptotic time limit is a clear indication of the break down of the ideal MHD model, which fails when very short scale perturbations are excited. Typically, the most relevant new (with respect to the ideal MHD model) dynamics that appear on short scales are associated with charge separation, i.e., with the finite parallel electric field fluctuations ($\delta E_{||}$) due to, e.g., finite ion Larmor radius ($l_i$), small but finite electron inertia and finite plasma resistivity. In the presence of finite $\delta E_{||}$, additional effects are to be expected also from wave-particle interactions, which yield collisionless wave dissipation known as Landau damping. Incorporating such “kinetic” effects essentially allows finite energy propagation across the resonant surfaces $x = x_{Re}$. Thus, we may expect that wave energy can no longer “pile up” at these radial locations and that, as a consequence, all wave-function singularities are removed on short scales. Here, we limit our discussion to the case in which $m_e/m_i \ll \beta_e \ll 1$ (with $\beta_e$
being the ratio between the electron kinetic and magnetic pressures), i.e., the electron thermal speed is much larger than the Alfvén velocity. Furthermore, for the sake of simplicity, we also assume \((k_x^2 + k_y^2)\rho^2_i \equiv k_{\perp}^2 \rho_i^2 \ll 1\). It is then possible to show that eq. (47) becomes [48, 49]

\[
\left\{ k_{\perp}^2 \rho^2_i \nabla^2 + \nabla \cdot \epsilon_{\perp} \nabla \right\} \hat{\xi}_{xt} = 0,
\]

where

\[
\rho_k^2 = \frac{1}{4} \left[ \frac{3}{4} (1 - i \delta_i) + \frac{T_e}{T_i} (1 - i \delta_e) \right] \rho_i^2 - \frac{1}{16 \omega^2}.
\]

Here, \(\delta_i\) and \(\delta_e\) indicate, respectively, ion and electron Landau damping contributions, whereas the term proportional to \(\eta\) is due to finite plasma resistivity (cf. eq. (10)). In eq. (59), the singularity at \(\epsilon_{\perp} = 0\) is clearly removed by the term including the 4-th order derivative, which is also proportional to \(\rho_k^2 \ll 1\), indicating the formation of a boundary layer around the shear Alfvén resonant surface. In fact, eq. (59) describes the mode-conversion of a long wavelength MHD mode (the shear Alfvén wave) to a short wavelength kinetic mode: the Kinetic Alfvén Wave (KAW) [8, 48, 50, 51]. The rate at which KAW’s are excited is exactly that of eq. (54). Thus, the resonant energy absorption rate of shear Alfvén waves may be interpreted as a power transfer to short wavelength modes, which, eventually, may be absorbed by the background plasma. Note, however, that resonant absorption of the MHD wave and KAW dissipation are mutually independent processes. Indeed, eq. (54) itself is independent on the details of the dissipation mechanism.

The WKB local dispersion relation of KAW’s is

\[
\omega^2 = \left( 1 + 4k_{\perp}^2 \rho_k^2 \right) \omega_{\perp}^2.
\]

Equation (61) indicates that KAW’s are propagating for \(\epsilon_{\perp} > 0\) and become cut-off for \(\epsilon_{\perp} < 0\).

### 2.3 Waves in general non-uniform axisymmetric equilibria

In the following, we derive the governing equations for shear Alfvén waves in general non-uniform plasma equilibria, characterized by symmetry under rotations about a given axis, which we assume to be the \(z\)-axis. In this system, we take \(a\) as the typical scale-length perpendicular to the equilibrium magnetic field, and \(R_0\) as the characteristic parallel scale-length. Such a choice is not a casual one, since it will allow us to naturally use it in the discussion of toroidal plasma equilibria in later sections; it also yields no restrictions of validity of our analysis, since it is a quite general one, until a specific ordering of \(a\) with respect to \(R_0\) is not further assumed.

In order to obtain the simplest, yet relevant, set of wave equations, we focus on high wave-number modes. More specifically, we assume waves of the form \(\propto \exp(i n \varphi)\), with \(\varphi\) being the trivial angular coordinate of the rotationally symmetric equilibrium, and \(n \gg 1\) being the toroidal mode number. In such a way, perpendicular (to the magnetic field \(B_0\)) wavelengths are typically \(\lambda_{\perp} \approx a/n\), whereas \(\lambda_{\parallel} \approx R_0\) in order to minimize the stabilizing influence of magnetic field line bending. Recollecting the results of sections 2.1 and 2.2, the present high-\(n\) assumption separates the typical time-scales of incompressible shear Alfvén waves and of fast magnetosonic (compressional) waves. The \(n \gg 1\) assumption will be maintained throughout this section, unless otherwise explicitly stated.

The governing equations for shear Alfvén waves can be derived from the quasi-neutrality condition \(\nabla \cdot \delta J = 0\); i.e.,

\[
\nabla \cdot \delta J_{\perp} + B_0 \cdot \nabla \left( \frac{\delta J_{\parallel}}{B_0} \right) = 0 ,
\]
where $\delta J$ is the perturbed current. The expression for $\delta J_\parallel$ is given by the parallel Ampère's law

$$4\pi \delta J_\parallel = c \hat{b} \cdot \nabla \times (\nabla \times \delta A) ,$$

$\delta A$ being the perturbed vector potential, and $\hat{b} \equiv B_0/B_0$. Thus, in the present axisymmetric equilibrium,

$$\delta J_\parallel = -\frac{c}{4\pi} \nabla_\perp^2 \delta A_\parallel \left(1 + O\left(\frac{a}{nR_0}\right)\right) .$$

Furthermore, $\delta A_\parallel$ can be expressed in terms of the scalar potential perturbation $\delta \phi$ via the parallel Ohm’s law $\delta E_\parallel = 0$ (see eq. (4)); i.e.,

$$-\hat{b} \cdot \nabla \delta \phi + \frac{i \omega}{c} \delta A_\parallel = 0 .$$

Note that time dependence of the form $\exp(-i \omega t)$ has been assumed here.

The perpendicular current perturbation is obtained from the perpendicular force balance (cf. eq. (2)); i.e.,

$$\delta J_\perp = -i \omega \frac{c}{B_0} B_0 \times \partial_0 \delta v_\perp + \frac{c}{B_0} B_0 \times \nabla \delta P +$$

$$\frac{J_\parallel}{B_0} \delta B_\perp - \frac{\delta B_\parallel}{B_0} \frac{c}{B_0} B_0 \times \nabla P_0 .$$

Here, $P_0$ is the equilibrium pressure and $\beta \equiv 8\pi P_0/B_0^2 \ll 1$. In eq. (64), the perpendicular velocity $\delta v_\perp$, meanwhile, is given by the Ohm’s law

$$B_0 \times \delta v_\perp = -c \nabla_\perp \delta \phi ;$$

and the pressure perturbation $\delta P$ by the equation of state, eq. (9),

$$-i \omega \delta P + \delta v_\perp \cdot \nabla P_0 \simeq 0 ,$$

corresponding to an incompressible plasma behaviour. Equation (66) gives

$$\delta P = \frac{c}{i \omega} \frac{B_0 \times \nabla \delta \phi}{B_0} \cdot \nabla P_0 \simeq \left(\frac{ck_\perp}{\omega B_0}\right) \frac{\partial P_0}{\partial r} \delta \phi ,$$

where $r$ is a radial coordinate, orthogonal to the local magnetic flux surface. From eq. (64), we then find

$$(1 + O\left(\frac{1}{n}\right)) \nabla \cdot \delta J_\perp = i \nabla \cdot \left[\frac{c^2}{B_0^2} \partial_0 \omega \nabla_\perp \delta \phi \right] - 2c \kappa \frac{B_0}{B_0^4} \frac{\partial P_0}{\partial r} \delta \phi =$$

$$c \frac{B_0 \times \nabla P_0}{B_0} \cdot \nabla_\perp \left(4\pi \delta P + B_0 \delta B_\parallel\right) ,$$

where $\kappa \equiv \vec{b} \cdot \nabla \vec{b}$ is the magnetic field curvature.

For $n \gg 1$, the fluctuations are localized in the radial direction, i.e., their typical width is much smaller than $a$, the perpendicular scale-length of the system. Moreover, the modes considered here are characterized by the shear Alfvén time scale, which is
much longer (an order $O(nR_0/a)$) than that of the compressional Alfven wave. As a consequence, the plasma behaves as an incompressible fluid, and

\[
4\pi \delta P + B_0 \delta B_\parallel = 0
\]

to the lowest order in $n$, which expresses the perpendicular pressure balance. Incidentally, we note that the decoupling from the compressional wave, at the lowest order in the high-$n$ limit, is only one of the reasons that cause this case to be one of particular relevance. In fact, as it will be shown in the following, shear Alfven waves may be driven unstable by their resonant interaction with MeV ions, which are present in the plasma, e.g., as fusion products.

The high-$n$ assumption, along with eq. (68), allows us to further simplify the expression of $\nabla \cdot \delta \mathbf{J}_\perp$. Combining eqs. (63) through (68), eq. (62) can be cast into the form of the following vorticity equation

\[
B_0 \mathbf{b} \cdot \nabla \left[ \frac{1}{B_0} \nabla_\perp^2 \mathbf{b} \cdot \nabla \delta \phi \right] + \nabla \cdot \left[ \frac{4\pi \rho_0}{B_0^3} \omega^2 \nabla_\perp \delta \phi \right] - 
\]

\[
-8\pi \kappa \times \frac{B_0}{B_0^2} \cdot \nabla_\perp \left[ \left( \frac{B_0 \times \nabla P_0}{B_0^2} \right) \cdot \nabla_\perp \delta \phi \right] = 0.
\]

Equation (69) is the general equation governing high-$n$ shear Alfven waves in low-$\beta$, non-uniform, axisymmetric plasma equilibria. Applied to different plasma configurations and geometries, it will be the starting point of our analyses of the general properties of shear Alfven wave spectra.

2.4. Waves in a cylinder. - The simplest plasma equilibrium (with cylindrical symmetry) we analyze, with the help of eq. (69), is a pressureless ($P = 0$) screw pinch. A screw pinch of length $2\pi R_0$ is characterized by an equilibrium magnetic field $\mathbf{B}_0 = (0, B_0\vartheta(r), B_0z(r))$, where $(r, \vartheta, z)$ is a cylindrical coordinate system. Shear Alfven waves in such a plasma equilibrium (which is a 1-D equilibrium as a sheared slab) are characterized by a continuous spectrum $\omega^2 = k_A^2(r)\vartheta_0^2(r)$. The only new feature introduced when considering a screw pinch is the presence of finite magnetic field curvature. If we assume to have a shear Alfven oscillation of the scalar potential $\delta \phi(r, \vartheta, z, t)$ of the form

\[
\delta \phi(r, \vartheta, z, t) = e^{in\vartheta/R_0-m\varrho} \delta \phi_{m,n}(r, t),
\]

equation (69) becomes, for $\delta \phi_{m,n}(r, t)$,

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^3 \left( n - m \frac{q(r)}{v_A^2} \right)^2 + r^3 \frac{R_0^2}{v_A^2} \frac{\partial^2}{\partial t^2} \right] \frac{\partial}{\partial r} \left( \frac{\delta \phi_{m,n}(r, t)}{r} \right) = 
\]

\[
\frac{m^2 - 1}{r^2} \left[ \left( n - m \frac{q(r)}{v_A^2} \right)^2 + \frac{R_0^2}{v_A^2} \frac{\partial^2}{\partial t^2} \right] \delta \phi_{m,n}(r, t) - 
\]

\[
\left( \frac{\partial}{\partial r} \frac{R_0^2}{v_A^2} \right) \frac{\partial^2}{\partial t^2} \left( \frac{\delta \phi_{m,n}(r, t)}{r} \right),
\]

with $q(r) = rB_0\vartheta(r)/(R_0B_0\vartheta(r))$ playing the role of the so-called safety factor in a tokamak, and $m$ and $n$ that of the poloidal and toroidal mode number respectively. Equation (71) is the continuity (or quasi-neutrality) equation $\nabla \cdot \delta \mathbf{J} = 0$, the terms $\approx \partial^2/\partial t^2$
coming from the divergence of the perpendicular polarization current, and the others from the divergence of the parallel current. Then eq. (71), solved as an initial value problem [52, 53], gives

\[
\delta \phi_{m,n} (r, t) \approx \frac{1}{t} e^{-i \omega_A (r) t}
\]

(72)

for the time asymptotic behaviour of \( \delta \phi_{m,n} (r, t) \), where

\[
\omega_A^2 (r) = \frac{v_A^2}{R_0} \left( n - \frac{m}{q (r)} \right)^2.
\]

(73)

If we identify \( k_\parallel (r) \equiv (n - m / q(r)) / R_0 \), the nature of the continuous shear Alfvén spectrum in a screw pinch is clear. Shear Alfvén waves, in fact, consist of “local” plasma oscillations, with frequency \( \omega^2 (r) = \omega_A^2 (r) \) satisfying eq. (73) and continuously changing throughout the plasma column. For the shear Alfvén continuous spectrum in a screw pinch the same considerations hold, which were made for a sheared slab in sect. 2.2.

The validity of eqs. (72) and (73) is general and it is not limited by the high-\( n \) assumption made in deriving eqs. (69) and (71). In fact, exploiting the local nature of continuum plasma oscillations, eq. (71) reduces to

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^3 \left( n - \frac{m}{q (r)} \right)^2 + r^3 \frac{R_0^2}{v_A} \frac{\partial^2}{\partial r^2} \right] \frac{\partial}{\partial r} \left( \frac{\phi_{m,n} (r, t)}{r} \right) = 0,
\]

(74)

which holds for arbitrary \((m, n)\) in the boundary layer where the continuous spectrum is excited. Such general validity is confirmed by the results of numerical simulations of the time asymptotic behaviour of a radially extended initial perturbation of the low-\( n \) Alfvén continuum in a cylindrical configuration. We make use of the initial-value version of the MARS code (see appendix sect. 5.2), and choose an equilibrium configuration with the safety factor \( q \) ranging from \( q = 1.1 \) in the centre to \( q = 1.85 \) at the edge of the plasma column and a flat plasma-density radial profile. This equilibrium corresponds to an Alfvén continuum, for a mode perturbation with poloidal and toroidal mode numbers \((m, n) = (1, 1)\), ranging from \( \omega / \omega_A \simeq 0.1 \) in the centre to \( \omega / \omega_A \simeq 0.45 \) at the edge of the plasma (here we define \( \omega_A \equiv v_A / R_0 \)). The assigned initial perturbation is shown in fig. 1, and consists of a radially extended velocity perturbation (here \( s \equiv (V (\psi) / V_{tot})^{1/2} \), see appendix sect. 5.2). As already shown in sect. 2.2, we expect the radial component of this perturbation to oscillate at the local Alfvén frequency, with the amplitude decaying asymptotically in time as \( \propto 1/t \), whereas the poloidal component asymptotically oscillates with constant amplitude. In fig. 2 the time dependence of the radial component of the perturbed velocity is shown, at a fixed value of the radial coordinate. The poloidal component of the perturbed velocity is shown in fig. 3. To better illustrate the phase mixing phenomenon, we also show in figs. 4 the profile of the perturbed radial velocity at several times. The initially smoothed and extended perturbation changes in time and becomes more and more jagged, losing phase correlation. In fact, each fluid element oscillates at the local Alfvén frequency, losing coherence with the motion of adjacent elements. This fact qualitatively explains the reason of the name “phase mixing”, which indicates this phenomenon. A careful analysis of the time behaviour of the velocity perturbation at a fixed value of the radial coordinate (e.g., making its Fourier transform in time) shows that a global oscillation exists, which is present at all the values of the radial coordinate and is superimposed to the local Alfvén frequency (see fig. 5). These “global” modes are, generally speaking, potentially dangerous if they are driven unstable,
Fig. 1. - Initial perturbation of the contravariant radial component of the perturbed velocity, $v_{I,1}^s$.

Fig. 2. - Time dependence of the contravariant radial component of the perturbed velocity, $v_{I,1}^s$, at $s = 0.2$. Also the envelope of the curve decaying as $\propto 1/t$ is shown.
Fig. 3. - Time dependence of the contravariant poloidal component of the perturbed velocity, \( v_1^{\phi} \), at \( s = 0.2 \).

Fig. 4. - Radial profile of the contravariant radial component of the perturbed velocity, \( v_r^{\phi} \), at several times.
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Fig. 5. – Alfvén frequency spectrum of the contravariant radial component of the perturbed velocity, \( v'_{\|} \), at several radial positions.

because they may affect large regions of the plasma and eventually yield confinement degradation for both particles and energy. The global mode observed in the frequency spectrum of fig. 5 is characterized by a purely real frequency, which lies just above the Alfvén continuum, and by a perturbed radial velocity almost uniform over the whole plasma cross section.

As a further example, it is instructive to analyze a cylindrical configuration with a plasma density decreasing with radius, as generally encountered in confined plasma equilibria. Here, we assume \( \rho_0(s) = \rho_0(0)(1 - s^4)^3 \). Moreover, a perturbation with \( (m, n) = (2, 1) \) is considered. In fig. 6 the corresponding Alfvén continuum is shown. It has to be noted that the Alfvén continuum frequency increases toward the edge of the plasma because of the corresponding density decrease. In fig. 6, a line corresponding to a Global Alfvén Eigenmode (GAE) is also shown, whose corresponding radial-velocity eigenfunction is reported in fig. 7. Several global Alfvén eigenmodes are in fact present with frequencies just below the continuous spectrum, which correspond to eigenfunctions with different radial modes.

Some insight in these global modes can be obtained, following refs. [46, 47] and expanding the quasi-neutrality equation, \( \nabla \cdot \delta J = 0 \), around an extremum, \( r_{\text{ext}} \), of the Alfvén spectrum \( (\omega_A'(r_{\text{ext}}) = 0) \). Defining \( F(\omega, r) \equiv \omega^2 - k^2_{\|} \Omega^2 \) and \( E \equiv -e\delta \phi_{m,n}/(B_0 r) \) (the poloidal component of the electric field), for consistency of notation with refs. [46, 47], the following differential equation for \( E \) is obtained

\[
\frac{d}{dr} \left( Fr \frac{d}{dr} (rE) \right) - m^2 FE - g(r)E = 0,
\]

where \( g(r) \) is given by [46, 47]

\[
g(r) = v_A^2(r_{\text{ext}}) \left[ \frac{r k_{\|}}{dr} \frac{d}{dr} \left( \frac{dk_{\|}}{dr} \right) - \frac{mr k_{\|}}{B_0} \frac{d}{dr} \frac{d}{dr} \left[ \frac{\psi_{eq}}{r_{\text{ext}}} \right] \right],
\]

and \( \psi_{eq} \) is the equilibrium poloidal magnetic field flux function (see appendix sect. 5.1). Equation (75) is valid for arbitrary \( (m, n) \) and, thus, cannot be directly derived from eq. (69). Indeed, the term \( g(r) \) represents the effect of the equilibrium current, which is negligible \( (O(1/n)) \) in the high-\( n \) limit.
Fig. 6. - Alfvén frequency spectrum for \( g_0(s) = g_0(0)(1 - s^4)^3 \). A perturbation with \((m, n) = (2, 1)\) is considered. Also a Global Alfvén Eigenmode (GAE) is shown (dashed line).

Fig. 7. - Radial profile of the contravariant radial component of the perturbed velocity, \( v_{2,1}^r \), for the GAE mode shown in fig. 6.
Upon expanding $F$ around $r_{\text{ext}}$,

$$
F = \omega^2 - \omega_A^2 \simeq \omega^2 - (\omega_A^2(r_{\text{ext}}) + \frac{1}{2} \omega_A''(r_{\text{ext}}) r^2) = \frac{\omega_A^2(r_{\text{ext}})}{L^2} (\Delta^2 - r^2),
$$

with $x \equiv (r - r_{\text{ext}})$, $\Delta^2 = L^2[\omega^2 - \omega_A^2(r_{\text{ext}})]/\omega_A^2(r_{\text{ext}})$ and $L^2 = 2\omega_A^2(r_{\text{ext}})/\omega_A''(r_{\text{ext}})$. Substituting the last expression into eq. (75) and evaluating all the other quantities at $r = r_{\text{ext}}$, we obtain

$$
\frac{d}{dy}(y^2 - 1) \frac{dE}{dy} - \hat{\Delta}(y^2 - 1) E + g_0 E = 0.
$$

Here $y = x/\Delta$ and $g_0 = g(r_{\text{ext}})$, whereas $\hat{\Delta} = \Delta^2(m/r_{\text{ext}})^2$ represents the eigenvalue of the problem.

Analyzing eq. (78) in Fourier space, a Schrödinger like equation can be obtained

$$
\frac{d^2 \Psi}{dz^2} + [\hat{\Delta} - V(z)] \Psi = 0,
$$

where $\Psi(z) = (1 + z^2)^{1/2} \hat{E}(z) \Delta^{1/2}$, and

$$
\hat{E}(p) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dy e^{-iyy} E(y).
$$

The effective potential $V(z)$ is given by

$$
V(z) = - \frac{g_0}{1 + z^2} + \frac{1}{(1 + z^2)^2}.
$$

For $g_0 \leq 0$ (see fig. 8), the potential $V(z)$ has a maximum at $z = 0$. Thus no localized modes can exist for any value of $\hat{\Delta}$ and only continuum modes are present (we remind that the mode equation in the Fourier conjugate space is being analyzed, and that modes localized in such space correspond to extended (global) modes in the real space and vice versa). For $g_0 \geq 2$, $V(z)$ has a minimum at $z = 0$, is negative everywhere and approaches zero as $z \to \pm \infty$. Thus $\hat{\Delta} - V(z) = 0$ has real solutions only for $\hat{\Delta} > 0$, corresponding to bound states. These modes are localized in the narrow potential well around $z = 0$. Thus in real space they are broad and represent the discrete GAE spectrum. For $0 < g_0 < 2$ the potential $V(z)$ has two minima; again it can be shown [54] that bound states (GAE’s) can exist only for $\hat{\Delta} < 0$. Taking into account that, for the case shown in fig. 6, $g_0 \approx 0.4$, the existence of a GAE in that case can be justified on the basis of the previous analysis.

In the presence of non-ideal terms, as, e.g., resistivity or finite Larmor radius effects, the expression for $V(z)$ in this simplified analysis, can be written as [46]

$$
V(z) = - \frac{g_0}{1 + z^2} + \frac{1}{(1 + z^2)^2} + \sigma(1 + z^2),
$$

where $\sigma$ depends on the particular non-ideal effects considered. Thus for large values of $z$, $V(z)$ behaves like the potential of a simple harmonic oscillator implying bound states for all values of $g_0$ (see fig. 9) and then, also, for positive values of $\hat{\Delta}$. This corresponds to the fact that, beyond GAE’s, other discrete, closely spaced, localized modes (the KAW’s) exist, which, as shown in sect. 2.2, replace the Alfvén continuous spectrum.
Fig. 8. - Potential profile for different values of $g_0$.

Fig. 9. - Potential profile for $\sigma = 0.1$ and $\sigma = 0$, with $g_0 = -1$. 
The GAE's are considered to be not so dangerous for tokamak plasmas because, in a two-dimensional (2-D) equilibrium, it can be shown that toroidicity generally acts on such modes as a stabilizing effect [55]. In fact, different poloidal harmonics are coupled together, as it will be shown in the following sections, and GAE's very easily suffer the so-called continuum damping.

2.5. Waves in a torus: analytic theory. — In toroidal plasma equilibria, the situation discussed so far changes qualitatively. The poloidal-symmetry breaking due to the toroidal field variation on a given magnetic flux surface causes different poloidal harmonics to be coupled. In contrast with the cylindrical case, $m$ is no longer a good quantum number. As a consequence, at the intersection of the cylindrical continuous spectra of two neighbouring poloidal harmonics, say $m$ and $m + 1$, the shear Alfvén continuum breaks up and a frequency gap appears [11] (cf. fig. 10). To demonstrate this, reconsider eq. (71) as modified by toroidicity:

$$
\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^3 \left( n - \frac{m}{q(r)} \right)^2 + r^3 \frac{R_0^2}{v_A^2} \frac{\partial^2}{\partial \theta^2} \right] \frac{\partial}{\partial r} \left( \frac{\delta \phi_{m,n}(r,t)}{r} \right) =
$$

$$
- \frac{\partial}{\partial r} \frac{R_0^2}{v_A^2} \frac{\partial^2}{\partial \theta^2} \frac{\partial}{\partial r} \left[ \delta \phi_{m+1,n}(r,t) + \delta \phi_{m-1,n}(r,t) \right] +
$$

$$
\frac{m^2 - 1}{r^2} \left[ \left( n - \frac{m}{q(r)} \right)^2 + \frac{R_0^2}{v_A^2} \frac{\partial^2}{\partial \theta^2} \right] \delta \phi_{m,n}(r,t) -
$$

$$
\left( \frac{\partial}{\partial r} \frac{R_0^2}{v_A^2} \frac{\partial^2}{\partial \theta^2} \right) \frac{\delta \phi_{m,n}(r,t)}{r} ,
$$

(82)

Equation (82) represents an infinite set of 1-D (in space) differential equations, which describe shear Alfvén waves coupled to (cut-off) compressional Alfvén waves [48, 56] in a 2-D (in space) toroidal equilibrium. Here, $\epsilon_0 = O(\epsilon)$ is the strength of the toroidal mode coupling in a toroidal equilibrium characterized by an inverse aspect ratio $\epsilon \equiv a/R_0$, with $a$ and $R_0$ the minor and major radius of the torus, respectively. Once fixed a frequency $\omega_0$, singular cylindrical poloidal harmonics $\delta \phi_{m_0,n}$ and $\delta \phi_{m_0+1,n}$, with $nq_0 = m_0 + 1/2$, are excited at the radial position $r_0$, such that $v_A^2/(4q_0^2 R_0^2) = \omega_0^2$. Here $v_A \equiv v_A(r_0)$ and $q_0 \equiv q(r_0)$. These components are thus the dominant ones near the considered surface and are described by the following coupled equations

$$
\frac{\partial}{\partial \delta q} \left[ \frac{\delta \omega}{2\omega_0} - \left( 1 - \frac{1}{2m_0 + 1} \right) n\delta q \right] \frac{\partial}{\partial \delta q} \delta \phi_{m_0,n} = - \frac{\epsilon_0}{4} \frac{\partial^2}{\partial \delta q^2} \delta \phi_{m_0+1,n} ,
$$

(83)

$$
\frac{\partial}{\partial \delta q} \left[ \frac{\delta \omega}{2\omega_0} + \left( 1 + \frac{1}{2m_0 + 1} \right) n\delta q \right] \frac{\partial}{\partial \delta q} \delta \phi_{m_0+1,n} = - \frac{\epsilon_0}{4} \frac{\partial^2}{\partial \delta q^2} \delta \phi_{m_0,n} ,
$$

which correspond to eq. (82) Fourier transformed (in time) and then linearized in $\delta q = q - q_0$ and $\delta \omega = \omega - \omega_0$. Note that the Fourier transform (in time) of $\delta \phi_{m_0,n}$ and $\delta \phi_{m_0+1,n}$ have been indicated with the same symbols for a simpler notation.

Equations (83) can be integrated once and give

$$
D(\delta \omega, n\delta q) \frac{\partial}{\partial \delta q} \delta \phi_{m_0,n} = B \left[ \frac{\delta \omega}{2\omega_0} + \left( 1 + \frac{1}{2m_0 + 1} \right) n\delta q \right] - \frac{\epsilon_0}{4} C ,
$$

(84)

$$
D(\delta \omega, n\delta q) \frac{\partial}{\partial \delta q} \delta \phi_{m_0+1,n} = C \left[ \frac{\delta \omega}{2\omega_0} - \left( 1 - \frac{1}{2m_0 + 1} \right) n\delta q \right] - \frac{\epsilon_0}{4} B ,
$$

where $B, C, D$ are constants.
with

\[
D(\delta \omega, n\delta q) = - \frac{\epsilon_0^2}{16} + \left[ \frac{\delta \omega}{2\omega_0} - \left( 1 - \frac{1}{2m_0 + 1} \right) n\delta q \right] \times \left[ \frac{\delta \omega}{2\omega_0} + \left( 1 + \frac{1}{2m_0 + 1} \right) n\delta q \right] .
\]

(85)

Here \( B \) and \( C \) are integration constants. In the \( \epsilon_0 \to 0 \) cylindrical limit, eqs. (84) give \( \delta \hat{\phi}_{m_0,n} \) singular at

\[
\delta \omega = 2\omega_0 n\delta q \left[ 1 - 1/(2m_0 + 1) \right] ,
\]

while the singularity in \( \delta \hat{\phi}_{m_0+1,n} \) occurs at

\[
\delta \omega = -2\omega_0 n\delta q \left[ 1 + 1/(2m_0 + 1) \right] .
\]

Thus, the local expression for the cylindrical shear Alfvén continuous spectrum is obtained.

When toroidicity is included (\( \epsilon_0 \neq 0 \)), both \( \delta \hat{\phi}_{m_0,n} \) and \( \delta \hat{\phi}_{m_0+1,n} \) are singular at

\[
D(\delta \omega, n\delta q) = 0 ,
\]

i.e.,

\[
n \delta q \left( 1 - \frac{1}{(2m_0 + 1)} \right) = \frac{\delta \omega}{2\omega_0 (2m_0 + 1)} \pm \sqrt{\frac{\delta \omega^2}{4\omega_0^2} - \frac{\epsilon_0^2}{16} \left( 1 - \frac{1}{(2m_0 + 1)^2} \right)} .
\]

(86)
If $4\delta \omega^2 \leq \epsilon_0^2 \omega_0^2 (1 - 1/(2m_0 + 1)^2)$, eq. (86) has no real solutions. Thus, it describes the formation of a frequency gap in the continuum, at the intersection of the $m_0$ and $m_0 + 1$ cylindrical harmonics, as shown in fig. 10. The width of the gap is

\[
\delta \omega = \epsilon_0 |\omega_0| \sqrt{1 - 1/(2m_0 + 1)^2}
\]

Within this frequency window, discrete shear Alfvén modes can exist, with normalizable well behaved eigenfunctions, characteristic of a discrete spectrum [12, 13]. These modes, known as Toroidal Alfvén Eigenmodes (TAE’s), are bound eigenstates of the system. Very roughly speaking, they exist because, at the intersection of the cylindrical continua, two modes, $m$ and $m + 1$, are degenerate and propagate in opposite directions along the magnetic field (since $k_{\parallel m+1,n} = -k_{\parallel m,n}$). Their beating generates a standing wave, which is further localized into a bound state by equilibrium inhomogeneities. Although the TAE poloidal harmonics are regular, the corresponding eigenfunctions are peaked at the gap positions and have a typical width $|n\delta q| \approx \epsilon_0$ (cf. fig. 11). The mechanism of toroidal coupling can, however, couple many poloidal harmonics, making the radial extent of the TAE much broader. In the absence of radial equilibrium variations of $\omega_A/q$, $\omega_A = v_A/R_0$ being the Alfvén frequency, the spectrum sketched in fig. 11 would be invariant under translations, and different poloidal harmonics would have the same (translated) shape. With equilibrium inhomogeneities, the TAE mode is further radially localized into a bound state and only a finite number of poloidal harmonics are coupled to give the TAE eigenmode structure. The amplitude of the poloidal harmonics outside the localization region rapidly (exponentially) decays away from it (cf. fig. 12(a)). To be somewhat more specific, we note that the central gap frequency $\omega_G^2 = \omega_A^2/(4q^2 R_0^2)$ varies with the radial position and a given TAE frequency is in the continuum forbidden window over a limited radial interval (cf. fig. 12(b)), $|n \delta q| = n \epsilon_0 |q^2 L A|$, $L_A$ being the scale length of $\omega_A^2/q^2 = v_A^2/(4q^2 R_0^2)$. This estimate also gives the typical number of coupled poloidal harmonics which are in the gap region.

At surfaces where the TAE frequency coincides with the local shear Alfvén continuum frequencies, the wave energy is resonantly absorbed [48, 56]. The mode is radially bound by a local potential well within the gap and has to tunnel through regions in which it decays in order to reach the continuous spectrum. Generally, the mode amplitude is, therefore, exponentially small at resonances and the wave absorption rate, proportional to the amplitude squared, is correspondingly small. Therefore, the TAE mode is undamped to the lowest order and attracts a significant attention, since it may be driven linearly unstable by energetic ions in the MeV range, which have typical velocities $v_H \approx v_A$. It is worthwhile to note that in this case the resonant wave absorption rate is proportional to the square of the amplitude of the magnetic field compression as in refs. [48, 56], but, unlike there, in the case under examination, toroidal coupling broadens the TAE mode width to a much larger extent than that of a single poloidal harmonic. Thus, it may be expected that, in the toroidal case, the resonant absorption rate, due to non-local coupling to the continuous spectrum, is larger than in the simpler cylindrical geometry.

In the case of TAE, resonantly excited by energetic particles, the high-$n$ assumption, made so far for mere simplicity of the model, takes up a more profound physical motivation. These modes can tap free energy from the energetic-particle source via the particle diamagnetic drifts $\omega_{\star P_\perp} = k_0(|e H_0|/\epsilon_H B_0)|\nabla \ln P_{\perp H}| \propto m$. In the following, it will be shown that $m \approx n q$; thus the energetic particle drive is expected to grow linearly with $n$ [20] until phase decorrelation of wave-particle resonances sets in at $k_{\perp} \rho_{LH} \approx k_r \rho_{LH} \approx 1$, with $\rho_{LH}$ the energetic-particle Larmor radius. Considering that $k_r \approx$
Fig. 11. - (a) Gap formation and gap structure for the poloidal harmonics $m$, $m + 1$ and $m + 2$. The form of each eigenfunction is given; in (b) for $\delta \phi_{m,n}$, in (c) for $\delta \phi_{m+1,n}$ and in (d) for $\delta \phi_{m+2,n}$.
Fig. 12. – (a) TAE eigenmode structure. The gap region $|n\delta q| \approx n\delta_0 |q/L_A|$ corresponds to the spatial interval in which the TAE frequency is in the forbidden frequency window of the continuous spectrum. The amplitude of the poloidal harmonics outside the TAE localization region is exponentially small. (b) Resonant excitation of the shear Alfvén continuum at the TAE frequency.
\( (1/\Delta r_{TAE}) = (nq'/\varepsilon_0) \) (cf. fig. 11), we can infer that the most unstable TAE mode number should be expected for \( n_{max} \approx (1/q)\varepsilon_0(a/\rho_{LH}) \). In present-day tokamaks, like TFTR [3] and JET [2], \( n_{max} \approx 1 \div 10 \), but in next-generation reactor-relevant devices like ITER [1] a range of \( n_{max} \approx 20 \div 60 \) is expected. For the high-\( n \) case is the physically relevant one to analyze, thus, eq. (69) can be considered the starting point of all the quantitative studies of TAE waves reported in this section.

2.5.1. Two-Dimensional WKB analysis

From now on, we use a toroidal coordinate system \((r, \vartheta, \varphi)\) for the description of a tokamak plasma equilibrium with major and minor radii respectively \( R_0 \) and \( a \) (cf. fig. 13). More precisely, \( r \) is a radial-like flux variable, \( i.e.\), \( B_0 \cdot \nabla r = 0 \), \( \vartheta \) is the poloidal angle and \( \varphi \) is the ignorable toroidal angle, on which the axisymmetric toroidal equilibrium does not depend. Details on the specific features of tokamak plasma equilibria and on their numerical computations are reported in appendix 5.1. For the present discussion we simply ought to recall that a tokamak plasma is characterized by an equilibrium magnetic field which is essentially oriented in the \( \varphi \) direction; \( i.e.\), \( B_{0\varphi} \gg B_{0\vartheta} \).

Typically, the most unstable shear Alfvén waves have \( k_{||} \approx (1/R_0) \) in order to minimize magnetic field line bending. Thus, we must have \( \mathbf{b} \cdot \nabla \delta \phi = O(1/R_0)\delta \phi \). Formally,

---

Fig. 13. - Toroidal coordinate system \((r, \vartheta, \varphi)\) for a tokamak plasma equilibrium with major radius \( R_0 \) and minor radius \( a \).
we may write the $\mathbf{b} \cdot \nabla$ operator as

$$
\mathbf{b} \cdot \nabla = \mathbf{b} \cdot \nabla \varphi \left[ \frac{\partial}{\partial \varphi} + \frac{\mathbf{B}_0 \cdot \nabla \vartheta}{\mathbf{B}_0 \cdot \nabla \varphi} \frac{\partial}{\partial \vartheta} \right],
$$

where, recalling that $B_0 \varphi \gg B_0 \vartheta$, $\mathbf{b} \cdot \nabla \varphi \equiv (1/R_0)$. Meanwhile, the most general Fourier representation we may use for $\delta \phi$ is, for a single $n$,

$$
\delta \phi(r, \vartheta, \varphi, t) = e^{i(n\varphi - \omega t + n \lambda(r, \vartheta))} \sum_m e^{-im\vartheta} \delta \hat{\phi}_{m,n}(r),
$$

where $\lambda(r, \vartheta)$ is a periodic function of $\vartheta$, which is chosen such that $\mathbf{b} \cdot \nabla \delta \phi = O(1/R_0) \delta \phi$. Using eqs. (88) and (89), we have

$$
e^{-i(n\varphi - \omega t + n \lambda(r, \vartheta) - m\vartheta)} \mathbf{b} \cdot \nabla \left[ e^{i(n\varphi - \omega t + n \lambda(r, \vartheta) - m\vartheta)} \delta \hat{\phi}_{m,n} \right] =
$$

$$
\frac{i}{R} \left[ n - \left( m - n \frac{\partial \lambda}{\partial \vartheta} \right) \frac{\mathbf{B}_0 \cdot \nabla \vartheta}{\mathbf{B}_0 \cdot \nabla \varphi} \right] \delta \hat{\phi}_{m,n}.
$$

Therefore, a possible choice for $\lambda(r, \vartheta)$, which yields $k_{||} \approx (1/R_0)$ on a given magnetic flux surface $r$, is given by

$$
\lambda(r, \vartheta) = - \left( \frac{m}{nq(r)} \right) \left( \int_0^\vartheta \frac{\mathbf{B}_0 \cdot \nabla \varphi}{\mathbf{B}_0 \cdot \nabla \vartheta} d\vartheta' - q(r) \vartheta \right),
$$

with the safety factor $q(r)$ defined as follows:

$$
q(r) = \frac{1}{2\pi} \oint \frac{\mathbf{B}_0 \cdot \nabla \varphi}{\mathbf{B}_0 \cdot \nabla \vartheta} d\vartheta.
$$

Indeed, with the present choice for $\lambda(r, \vartheta)$, we have

$$
e^{-i(n\varphi - \omega t + n \lambda(r, \vartheta) - m\vartheta)} \mathbf{b} \cdot \nabla \left[ e^{i(n\varphi - \omega t + n \lambda(r, \vartheta) - m\vartheta)} \delta \hat{\phi}_{m,n} \right] =
$$

$$
\frac{i}{qR} (nq - m) \delta \hat{\phi}_{m,n},
$$

which yields $k_{||} \approx (1/R_0)$ for a mode with a radial localization, $\Delta(nq(r))$, such that $(nq - m) = O(1)$. Thus, in the present representation, we generally have $nq \approx m$ and a radial localization (for a single Fourier mode) $\Delta r_{m,n} \approx 1/(nq')$, which implies $(1/k_{||}) \approx 1/(nq')$ to the lowest order in $n$ (cf. fig. 11).

The high-$n$ approximation allows us to assume that the 2-D TAE eigenmode structure is characterized by two separate spatial scales (cf. fig. 11): a scale $\approx \Delta r_\perp \equiv 1/(nq')$ describing the fast radial variation of each poloidal harmonic of the TAE wave function, and a scale $L_A \approx a \gg \Delta r_\perp$ accounting for slow equilibrium variations. This fact can be formally stated assuming that the poloidal harmonics $\delta \hat{\phi}_{m,n}$ can be written in the form [37]

$$
\delta \hat{\phi}_{m,n}(r) = A_j \phi(j, \zeta - j),
$$

where $\zeta \equiv nq - m_0$ is a newly defined radial (flux) variable, $m_0$ is a “central” poloidal mode-number and the $(\zeta - j)$ variable contains the fast radial variation of the poloidal eigenfunctions, while the slow variation due to equilibrium inhomogeneities is described by the slow scale of the discrete variable $j \equiv m - m_0$. Note that, here, the concept of $m_0$ as a “central” poloidal mode-number is well-defined since (cf. fig. 12(b)) the number
of poloidal harmonics which are effectively coupled together to form the TAE eigenmode structure is \( \Delta m \approx n c_0 \ll m \approx m_0 \). Using the two-scale argument further, we can rewrite eq. (94) as

\[
\delta \hat{\phi}_{m,n}(\zeta) = A(\zeta) \phi(\zeta, \zeta - j),
\]

where now \( \zeta \), rather than the discrete variable \( j \), contains the slow variation due to equilibrium inhomogeneities. Moreover, if we further take the Fourier transform of the function \( \phi(\zeta, \zeta - j) \) with respect to the fast variable \( (\zeta - j) \), we get [37]

\[
(95) \quad \delta \hat{\phi}_{m,n} = A(\zeta) \int_{-\infty}^{+\infty} d\theta \ e^{-i(\zeta-j)\theta} \Phi(\zeta, \theta).
\]

Here, \( \theta \) is the coordinate usually referred to as the extended poloidal angle in the theory of ballooning modes [57].

The two radial scale-lengths are naturally separated in eq. (95), where \( \theta \approx 1 \) reflects the fast scale \( \Delta r_s \approx 1/(n q') \), while the slow radial equilibrium variations are accounted for by the \( \zeta \) dependencies of the envelope \( A(\zeta) \) and the form function \( \Phi(\zeta, \theta) \). The Fourier representation of eq. (89), thus, becomes

\[
(96) \quad \delta \phi(r, \theta, \varphi, t) = e^{i(n \varphi + n \lambda(r, \theta) - m_0 \theta - \omega t)} A(\zeta) \sum_j e^{-i j \theta} \int_{-\infty}^{+\infty} d\theta \ e^{-i(\zeta-j)\theta} \Phi(\zeta, \theta).
\]

This equation is still exact since the existence of two scale-lengths in the problem is taken into account only when the WKB ansatz [36, 37]

\[
(97) \quad A(\zeta) \propto \exp \left( \int \theta_k(\zeta') d\zeta' \right)
\]

is explicitly made, along with the optimal ordering

\[
| \partial_k \ln \theta_k(\zeta) | \sim | \partial_k \ln \Phi(\zeta, \theta) | \sim (1/n),
\]

expressing \( \theta_k(\zeta) \) and \( \Phi(\zeta, \theta) \) slow dependencies on equilibrium profiles.

In order to make further analytical progress in solving eq. (69), we limit our investigation to a toroidal plasma equilibrium with circular shifted magnetic flux surfaces. Such an equilibrium can be completely described by two local plasma parameters [57]: the magnetic shear \( s \equiv r q'/q \) and the dimensionless pressure gradient \( \alpha \equiv -q^2 R_0 \beta' = -q^2 R_0 (8 \pi P_0'/B_0^2) \). In this case the strength of the toroidal mode coupling \( c_0 \) takes the explicit form \( c_0 \equiv 2(r_0/R_0 + \Delta') \), with \( \Delta' \) being the radial derivative of the so-called Shafranov shift \( \Delta(r) \), defined in the appendix sect. 5.5. Substituting eq. (96) along with \( A(\zeta) \) given by eq. (97) into eq. (69), we have

\[
\frac{\partial}{\partial \theta} \left( 1 + I^2 \right) \frac{\partial}{\partial \theta} + \Omega^2 \left( 1 + I^2 \right) \left( 1 + 2c_0 \cos \theta \right) + 
\alpha \left( \cos \theta + I \sin \theta \right) \Phi(\zeta, \theta) = 0,
\]

with

\[
I = \dot{s} \left( \theta - \theta_k + i \frac{\partial}{\partial \zeta} \right) - \alpha \sin \theta.
\]
Here, the boundary conditions are $\Phi(\zeta, \theta) \to 0$ as $|\theta| \to \infty$. In eq. (99), $\partial_k$ acts on the slow variation of $\Phi(\zeta, \theta)$ and $\theta_k(\zeta)$ only, the radial derivative on $A(\zeta)$ being taken care of by $\theta_k(\zeta)$ itself. Furthermore, $\Omega^2 \equiv \theta_k^2/\omega_k^2$. Thus, eq. (98) is completely equivalent to eq. (69), but it is written in a form which may be exploited for a two-scale WKB asymptotic expansion, by formally assuming $\partial_k = O(1/n^\tau)$, where either $\tau_T = 1/2$ or $\tau_T = 1/3$, as it will be shown shortly. Asymptotic expansions may be written for $\Phi(\zeta, \theta)$ and $\theta_k(\zeta)$, i.e.,

$$\Phi(\zeta, \theta) = \Phi^{(0)}(\zeta, \theta) + \Phi^{(1)}(\zeta, \theta) + \ldots,$$

$$\theta_k(\zeta) = \theta_k^{(0)}(\zeta) + \theta_k^{(1)}(\zeta) + \ldots,$$

where $\theta_k^{(0)} = O((1/n^\tau), i = 0, 1, 2, \ldots$. In the lowest order, the equation for $\Phi^{(0)}(\zeta, \theta)$ will read as eq. (98) with $I$ replaced by $I_0$, where

$$I_0 = \dot{s} \left( \theta - \theta_k^{(0)} \right) - \alpha \sin \theta.$$

The solution of the 1-D second order differential equation with homogeneous boundary conditions, i.e., $|\Phi^{(0)}(\zeta, \theta)| \to 0$ as $|\theta| \to \infty$, gives the local dispersion relation

$$F(\theta_k^{(0)}, \Omega^2, \alpha, \dot{s}) = 0.$$

Hence, this first step exactly reproduces the well known results of ballooning theory [57]. In eq. (101), the dependencies on $\Omega^2$, $\alpha$ and $\dot{s}$ have been indicated explicitly, since through them the dependence of $\theta_k^{(0)}$ on $\zeta$ takes place. Equation (101) can be viewed as the implicit form of the WKB trajectories in the $(\theta_k^{(0)}, \zeta)$ plane, which determines $\theta_k^{(0)}(\zeta)$. In the next order, it can be shown that

$$\theta_k^{(1)} = \frac{i}{2} \frac{d}{d\zeta} \ln \frac{\partial F}{\partial \theta_k^{(0)}};$$

leading to the following WKB expression for $A(\zeta)$

$$A(\zeta) = \frac{\text{const}}{\sqrt{\partial F / \partial \theta_k^{(0)}}} \exp i \int_{\zeta'}^{\zeta} \theta_k^{(0)} d\zeta'.$$

The WKB expression for the radial envelope becomes invalid close to the WKB turning points, identified by the condition $\partial F / \partial \theta_k^{(0)} = 0$. So, close to a turning point, we must recall the actual operator nature of $I$ in eq. (99), and solve the corresponding complete problem

$$F \left( -i \frac{\partial}{\partial \zeta}, \Omega^2, \alpha, \dot{s} \right) A(\zeta) = 0$$

in order to connect WKB solutions of different validity regions. Locally, within a small distance from the turning point position $(\theta_k^{(0)}, \zeta_T)$, it is always possible to expand eq. (104) as

$$A(\zeta) = 0.$$

$$A(\zeta) = \left[ \frac{1}{2} \frac{\partial^2 F}{\partial \theta_k^{(0)} \partial \zeta} \right]_{\zeta = \zeta_T} \left( -i \frac{\partial}{\partial \zeta} - \theta_k^{(0)} \right)^2 + \left( F(\theta_k^{(0)}, \Omega^2, \zeta) - F(\theta_k^{(0)}, \Omega^2, \zeta_T) \right) A(\zeta) = 0.$$
Thus, recalling that $\partial \zeta = O(1/n)$ when acting on equilibrium quantities, it is readily demonstrated that, generally, $\tau_T = 1/3$, whereas $\tau_T = 1/2$ if $\partial \zeta F(\theta_k^{(0)}, \Omega^2, \zeta) = 0$.

In general, eq. (105) is exactly solvable in terms of special functions or classic polynomials, depending on the order of the considered turning point. The connection of WKB solutions for the radial envelope from different WKB validity regions, with boundary conditions $A(\zeta) \rightarrow 0$ as $|\zeta| \rightarrow \infty$, leads to the global dispersion relation for the undamped TAE modes, in the form of a corresponding quantization condition:

$$
\oint \zeta(\theta_k^{(0)}|\Omega^2) d\theta_k^{(0)} = 2\pi (\mathcal{N} + \nu),
$$

where $\zeta(\theta_k^{(0)}|\Omega^2)$ satisfies the local dispersion relation $F(\theta_k^{(0)}, \Omega^2, \zeta) = 0$. Furthermore, $\mathcal{N}$ is the radial mode number, whereas $\nu = 1$ for WKB ray oscillations in the $(\theta_k^{(0)}, \zeta)$ phase space and $\nu = 0$ for phase space rotations [36]. The detailed solutions of eq. (106) is beyond the scope of the present work. Here, we will refer to refs. [26, 36, 49, 58], where eq. (106) is solved and discussed in great detail.

### 2.5.2. Gap formation and continuum damping

The lowest order eq. (98) can serve clarifying the existence of TAE modes as discrete (bound) eigenstates, well separated from the shear Alfvén continuous spectrum. Consider, for simplicity, the case $\alpha = 0$, $\theta^{(0)} = 0$. If we define $\Phi = \sqrt{1 + \epsilon_0^2 \theta^2} \Phi$, eq. (98) reduces to

$$
\left[ \frac{\partial^2}{\partial \theta^2} + \Omega^2 (1 + 2\epsilon_0 \cos \theta) - \frac{s^2}{(1 + s^2 \theta^2)^2} \right] \Phi = 0.
$$

In the large-$\theta$ limit, this is the Schrödinger equation of a free particle moving in a small perturbing periodic potential. For $\epsilon_0 = 0$ the solutions would be just plane waves $\Phi = \exp(\pm ik\theta)$ with $k = \Omega$. The traveling waves can be reflected by the background perturbation, but, in a 1-D periodic lattice of period $L$, they constructively combine in standing waves only when the Bragg reflection condition is met:

$$
2L = l\lambda, \quad \lambda \equiv \frac{2\pi}{k}, \quad l = 1, 2, 3, \ldots
$$

In this case $L = 2\pi$. Then we must have $k = \Omega = l/2$ for standing waves $\approx \sin(l\theta/2)$ and $\approx \cos(l\theta/2)$ to form. For the two lowest energy states, the eigenfrequency is given by

$$
\sin \left( \frac{\theta}{2} \right) : \quad \Omega^2 = \frac{1}{4} - \frac{\langle \sin \left( \frac{\theta}{2} \right) \left| \left. 2\epsilon_0 \Omega^2 \cos \theta \right| \sin \left( \frac{\theta}{2} \right) \rangle}{\langle \sin \left( \frac{\theta}{2} \right) \left| \left. \sin \left( \frac{\theta}{2} \right) \right\rangle} = \frac{1}{4} (1 + \epsilon_0),
$$

$$
\cos \left( \frac{\theta}{2} \right) : \quad \Omega^2 = \frac{1}{4} - \frac{\langle \cos \left( \frac{\theta}{2} \right) \left| \left. 2\epsilon_0 \Omega^2 \cos \theta \right| \cos \left( \frac{\theta}{2} \right) \rangle}{\langle \cos \left( \frac{\theta}{2} \right) \left| \left. \cos \left( \frac{\theta}{2} \right) \right\rangle} = \frac{1}{4} (1 - \epsilon_0),
$$

with $\langle f \left| g \right. \rangle = (1/4\pi) \int_0^{4\pi} d\theta fg^*$. It is not surprising that upper and lower edges of the shear Alfvén continuous spectrum are found in this way. The $\approx \sin(\theta/2)$ and $\approx \cos(\theta/2)$
eigenfunctions are, however, not normalizable. In other words, they are still eigenstates of the continuum. This draws a perfect analogy between Alfvén waves in a toroidal plasma and wave packets of Bloch electrons in a 1-D crystal lattice. The role of magnetic shear in eq. (107) is to create a ‘local’ potential well, which can transform TAE modes into discrete modes, with well behaved normalizable eigenfunction. The potential well is local because it varies with the radial position along with \( \theta_k^{(0)}(\zeta) \). It remains an effective well throughout the continuum forbidden frequency window (cf. fig. 12) and, of course, stops to be such when unnormalizable(1) eigenfunctions are excited in the continuous spectrum. The typical radial TAE localization region corresponds to \( \zeta \) values such that the WKB phase trajectories \( \theta_k^{(0)}(\zeta) \) are given by the local dispersion relation eq. (101), for purely real \( \theta_k^{(0)} \). Under normal circumstances, the localization region is contained within the gap, and the mode exponentially decays before it resonantly excites the continuum. Thus, any asymptotic expansion always neglects exponentially small terms, eq. (106) does not generally include damping. Perturbation analyses, based on energy conservation arguments, yield the following expression for TAE’s continuum damping [37, 36, 38]:

\[
\frac{\gamma}{|\omega|} = -\frac{1}{4 \sqrt{2} \alpha} \left\{ \Gamma_{\ell}(\hat{s}, \alpha) e^{-2|nq' L_A|/\omega \ell} + \Gamma_{u}(\hat{s}, \alpha) e^{-2|nq' L_A|/\omega u} \right\}.
\]

(110)

Here, the functions \( \Gamma_{\ell}(\hat{s}, \alpha) \) and \( \Gamma_{u}(\hat{s}, \alpha) \) represent, respectively, the absorption rate at the lower and upper shear Alfvén continuum. Meanwhile, \( \ell(\hat{s}, \alpha) \) and \( u(\hat{s}, \alpha) \) are the “tunneling” factors that describe the TAE wave cut-off while propagating towards the lower and upper continuum. All these functions have been calculated numerically for arbitrary values of \( \hat{s}, \alpha \) [36, 38].

The effect of pressure on the magnitude of continuum damping is important in that it may cause the tunneling region towards the lower continuous spectrum to disappear if \( \alpha \) exceeds a critical threshold \( \alpha_c(\hat{s}) \) [36, 38, 59], such that \( \ell(\hat{s}, \alpha_c(\hat{s})) = 0 \) [36, 38, 59], leading to a strong TAE coupling to the continuum and to a strong \( |\gamma/\omega| = O(\alpha) \) mode damping. In fact, it may be shown that the r.h.s. of eq. (110) diverges as \( \alpha \to \alpha_c(\hat{s}) \) [36, 38], due to the break-down of the perturbative assumption which yields that expression. In the \( |\hat{s}|, |\alpha| \ll 1 \) limit, the functions appearing in eq. (110) may be given an analytical expression [36, 37, 38]

\[
\Gamma_{\ell}(\hat{s}, \alpha) \simeq 2^{-1/2} |\hat{s}|^{-1} \pi^{-3/2} \left[ (1 - \hat{\alpha})^2 - \hat{\eta}^2 \right]^{-1/2} \left[ \text{arccosh} \left( \frac{1 - \hat{\alpha}}{\hat{\eta}} \right) \right]^{-3/2},
\]

(111)

\[
\Gamma_{u}(\hat{s}, \alpha) \simeq 2^{-1/2} |\hat{s}|^{-1} \pi^{-3/2} \left[ (1 + \hat{\alpha})^2 - \hat{\eta}^2 \right]^{-1/2} \left[ \text{arccosh} \left( \frac{1 + \hat{\alpha}}{\hat{\eta}} \right) \right]^{-3/2},
\]

(112)

\[
\ell(\hat{s}, \alpha) \simeq \frac{\hat{s}^2 \pi^2}{8} \text{arccosh} \left( \frac{1 - \hat{\alpha}}{\hat{\eta}} \right) \left[ (1 - \hat{\alpha})^2 + \hat{\eta}^2 \right]^{-1/2},
\]

(113)

\[
\ell(\hat{s}, \alpha) \simeq 2 \text{arccosh} \left( \frac{\hat{\alpha}}{\hat{\eta}} + \frac{2}{\pi |\hat{s}| \hat{\eta}} \right),
\]

(114)

(1) Incidentally, it is worthwhile to note that unnormalizable eigenfunctions in the ballooning \( \theta \) space correspond to singular eigenfunctions in real space.
with the definitions \( \tilde{\eta} = (1 + 1/|\tilde{s}| \exp(-1/|\tilde{s}|) \) and \( \tilde{\alpha} = \alpha (1 + \tilde{s})/\tilde{s}^2 \). From eq. (113), it follows [36, 38, 59]

\[
\alpha_c(\tilde{s}) = \frac{\tilde{s}^2}{1 + \tilde{s} \left[ 1 - (1 + 1/|\tilde{s}|) e^{-1/|\tilde{s}|} \right]}
\]

Thus, as \( \alpha \to \alpha_c(\tilde{s}) \),

\[
(115) \quad \frac{\gamma}{|\omega|} \approx -0.0037 \epsilon_0 (\eta_0 \epsilon_0 |\tilde{s}|)^{-3/2} \frac{\tilde{s}^2 (1 + |\tilde{s}|)}{(1 + \tilde{s})^2} \frac{e^{-1/|\tilde{s}|}}{(\alpha_c(\tilde{s}) - \alpha)^2} \left( \frac{r_0}{L_A} \right)^{3/2},
\]

where \( r_0 \) is the radial location of the high-\( n \) localized TAE mode. Equation (115) typically gives \( |\gamma/\epsilon_0 \omega| \approx 0.1 \div 1 \).

### 2.5.3. Excitation of kinetic TAE

As it was discussed in sect. 2’2, when the Alfvén continuum is resonantly excited the energy absorption at the resonant layer corresponds to mode conversion to KAW’s [48, 56]. In this case, and similarly to what happens for a TAE, two degenerate kinetic waves can form a standing wave: the Kinetic Toroidal Alfvén Eigenmode (KTE) [30]. As already discussed for eq. (59), kinetic effects modify eq. (98) into [49, 60]

\[
(16) \quad \left[ \partial_\theta (1 + \tilde{I}_0^2) \partial_\theta + \Omega^2 (1 + \tilde{I}_0^2) (1 + 2\epsilon_0 \cos \theta) + \alpha (\cos \theta + I_0 \sin \theta) - 4\Omega^2 (1 + \tilde{I}_0^2) \tilde{s}^2 (\theta - \theta_k^{(0)})^2 \rho_K^2 \right] \Phi(\zeta, \theta) = 0,
\]

where \( I_0 \) is given by eq. (100) and \( \rho_K^2 \equiv k^2_0 \rho_K^2 \), with \( \rho_K^2 \) given by eq. (60). It may be shown that, close to the accumulation points of the continuous spectrum (the gap boundaries), the solution of eq. (116) may be written as

\[
(117) \quad \Phi(\zeta, \theta) = (1 + \tilde{I}_0^2)^{-1/2} \left[ A_c(\tilde{\theta}) \cos \left( \frac{\tilde{\theta} + \theta_k^{(0)}}{2} \right) + A_s(\tilde{\theta}) \sin \left( \frac{\tilde{\theta} + \theta_k^{(0)}}{2} \right) \right],
\]

where \( \tilde{\theta} \equiv \theta - \theta_k^{(0)} \). Furthermore, \( A_c(\tilde{\theta}) \) and \( A_s(\tilde{\theta}) \) satisfy the coupled equations [49, 60]

\[
(118) \quad \frac{dA_c(\tilde{\theta})}{d\tilde{\theta}} = \left( \Omega^2 - \frac{1}{4} - \epsilon_0 \Omega^2 - \tilde{s}^2 \rho_K^2 \tilde{\theta}^2 \right) A_c(\tilde{\theta}),
\]

\[
\frac{dA_s(\tilde{\theta})}{d\tilde{\theta}} = \left( \Omega^2 - \frac{1}{4} - \epsilon_0 \Omega^2 - s^2 \rho_K^2 \tilde{\theta}^2 \right) A_s(\tilde{\theta}).
\]

Equation (118) is easily solved in proximity of lower and upper accumulation point of the shear Alfvén continuum and it yields, respectively,

\[
\Omega^2 = \frac{1 - \epsilon_0}{4} - i(2\mathcal{N} + 1) \left( \frac{2}{\epsilon_0} \right)^{1/2} \tilde{s} \rho_K,
\]

\[
(119) \quad \Omega^2 = \frac{1 + \epsilon_0}{4} + (2\mathcal{N} + 1) \left( \frac{2}{\epsilon_0} \right)^{1/2} \tilde{s} \rho_K,
\]

where \( \mathcal{N} \) is the radial mode number. The asymmetry between lower and upper KTE spectra is due to the KAW dispersion relation, eq. (61), which predicts propagating waves
for $\omega^2 > \omega_{A,m}^2$ (with $\omega_{A,m}^2 = v_A^2(n - m/q)^2/R_0^2$ being the local shear Alfvén frequency of the $m$-th poloidal harmonic) and wave cut-off for $\omega^2 < \omega_{A,m}^2$. In the case of the upper KTAE branch, the two degenerate KAW’s form a standing wave at the frequency-gap radial position, whereas, for the lower KTAE branch, KAW’s carry energy away from the mode localization region [30] (see fig. (14)). In fact, in the dissipationless limit,

![Diagram of propagating and cut-off regions of KAW's](image)

Fig. 14. – Sketch of propagating and cut-off regions of KAW’s.

eq. (60) yields purely real $\rho_K$ ($\delta_i = \delta_e = \eta = 0$) and eqs. (119) indicate that upper KTAE’s are undamped waves, while lower KTAE’s are affected by convective damping. The asymmetry between lower and upper KTAE spectra disappears when only the effect of finite plasma resistivity is considered in the definition of $\rho_K^2$ ($\rho_i = 0$): both the upper and lower branches are damped modes, located on straight lines in the complex frequency plane, emerging at $45^\circ$ from the upper and lower limits of the gap, respectively. This case was originally investigated in ref. [12], where purely “resistive” KTAE were called Resistive Periodic Shear Alfvén Eigenmodes (RPSAE’s).

### 25.4. Resonant excitation of gap modes

That shear Alfvén wave may be driven unstable by resonant interactions with MeV energetic ions was originally proposed in ref. [8]. This is due to the fact that, as stated before, for such particles the typical speed is $v_H \approx v_A$ and that the rate at which the mode taps the free energy source associated with equilibrium pressure gradients, $\omega_{eP,i}$, typically overcomes collisionless dissipation in velocity space (Landau damping $\propto \omega$), i.e., $\omega_{eP,i} > \omega$. On the contrary, both ions of the thermal plasma and electrons usually provide mode damping, since $\omega_{eP,i,e} < \omega$.

Resonant excitations of both TAE and KTAE may be analyzed only via kinetic description of energetic particle dynamics. It is possible to show that, with the contribution
of energetic ions, eq. (116) is modified into [26, 49, 60]

\[ \left[ \partial_t + \left( 1 - I_0^2 \right) \partial_x + \Omega^2 \left( 1 + I_0^2 \right) \right] \left( 1 + 2v_0 \cos \theta \right) + \alpha (\cos \theta + I_0 \sin \theta) - \]

\[ 4 \Omega^2 \left( 1 + I_0^2 \right) s^2 (\theta - \theta_K^2) \mu H = \frac{4 \pi \omega e H^2 q^2 R_0^2}{c^2} k_\theta^2 \left( \omega_{dH} \delta G_H \right), \]

where \( \langle . . \rangle = \int \langle . . \rangle d^3 v, \omega_{dH} = \Omega_{dH} (\cos \theta + I_0 \sin \theta) \) is the magnetic drift frequency, \( \Omega_{dH} = k_\theta \left( v_x^2 / 2 + v_y^2 \right) / (R_0 \Omega_H) \), \( \Omega_H = eH B_0 / (m_H c) \) and \( \delta G_H \) is the non-convective response of the energetic particle fluctuating distribution function \( \delta F_H \) to the wave fields, \( i.e., \delta G_H = \delta F_H + (i / \omega) \left( c / B_0 \right) (\delta \mathbf{E} \times \mathbf{b}) \cdot \nabla F_{H0} \), with \( F_{H0} \) being the energetic particle equilibrium distribution function \( \langle . . \rangle \) (c.f. appendix 5'5). In the limit \( k_H^2 \rho_{KH}^2 \ll 1 \) (with \( \rho_{KH} \equiv v_H / \Omega_H) \), the governing equation for \( \delta G_H \) can be shown to be [20] (see eq. (204) of appendix 5'5):

\[ \left[ \omega_t \partial_t + i (\omega - \omega_d) \right] \delta G_H = \frac{i eH}{m_H} QF_{H0} \frac{\omega_{dH}}{\omega} \Phi, \]

where \( \omega_t = v_t / (q R) \) is the transit frequency and \( QF_{H0} = (2 \omega \partial_{t,v} + k \times \mathbf{b} / \Omega_H \cdot \nabla) F_{H0} \). Although eq. (120) has been derived in the high-\( n \) limit, it also holds for low-\( n \) modes in the thin boundary layer at the gap region \( \tilde{\theta} \approx 1 / \epsilon_0 \).

Numerical investigations of TAE and KTAE resonant excitations by energetic ions will be thoroughly reported in sect. 2.6.3. Analytical solutions of eq. (120), instead, have been discussed in detail in refs. [26, 41, 49, 60]. Such detailed discussions are beyond the scope of the present work and, thus, we refer the interested reader to the cited literature. Here, we recall those results in some simple cases, considering circulating energetic ions only, \( i.e., \) particles which are never trapped between magnetic mirror points.

Two distinct scales determine the large \( |\tilde{\theta}| \) behavior of the solution of eq. (120) in the gap region: \( \tilde{\theta} \approx 1 \) and \( \tilde{\theta} \approx 1 / \epsilon_0 \). It is then possible to show that \( \Phi(\zeta, \tilde{\theta}) \) may be written in the form of eq. (117) [12], where \( A_\pm (\tilde{\theta}) \) and \( A_\mp (\tilde{\theta}) \) satisfy the following coupled equations [61], \( i.e., \) an extension of eqs. (118):

\[ \frac{d}{d \tilde{\theta}} A_\pm = \left( \Gamma_\pm - \beta - 2 s^2 \rho_{KH}^2 \tilde{\theta}^2 s^2 \rho_{KH}^2 \tilde{\theta}^2 \right) A_\pm + s \tilde{\theta} \delta A_\mp, \]

\[ \frac{d}{d \tilde{\theta}} A_\mp = - \left( \Gamma_\pm - \beta - 2 s^2 \rho_{KH}^2 \tilde{\theta}^2 s^2 \rho_{KH}^2 \tilde{\theta}^2 \right) A_\mp - s \tilde{\theta} \delta A_\pm, \]

where \( \Gamma_\pm = \Omega^2 / 4 \pm \epsilon_0 \Omega^2 / 20 \). Assuming, for the sake of simplicity, the case of an equilibrium distribution function \( F_{H0} \) symmetric in \( v || \), the following expressions can be obtained for the constants \( \beta, \dot{\delta} \) and \( \rho_{KH}^2 \), containing the contribution of the energetic particles:

\[ \dot{\beta} = \frac{2 \pi q^2}{B_0^2} R_0 \left( \frac{m_H}{\Omega} \left( \frac{v_x^2}{2} + v_y^2 \right) \right)^2 \left[ \frac{QF_{H0}}{(3 v || - 2 q R_0 \omega)} - \frac{QF_{H0}}{(3 v || - 2 q R_0 \omega)} \right], \]

\[ \dot{\delta} = - \frac{3 \pi q^3}{B_0^2} \frac{k_\theta}{\omega} \left( \frac{m_H}{2 \Omega} \right) \left( \frac{v_x^2}{2} + v_y^2 \right)^3 \left[ \frac{QF_{H0}}{(3 v || - 2 q R_0 \omega)} - \frac{QF_{H0}}{(3 v || - 2 q R_0 \omega)} \right], \]

\[ \rho_{KH}^2 = \frac{m_H}{2 B_0^2} \frac{1}{\Omega} \left( \frac{v_x^2}{2} + v_y^2 \right) \left[ \frac{QF_{H0}}{(3 v || - 2 q R_0 \omega)} - \frac{QF_{H0}}{(3 v || - 2 q R_0 \omega)} \right]. \]
The conditions for the excitation of TAE’s or KTAE’s can be easily seen from eqs. (122) [60]. For toroidal equilibrium terms ($\Gamma_-$ and $\Gamma_+$) sufficiently strong to dominate the kinetic terms, the “regular” MHD TAE mode prevails. This occurs as long as $|s^2(\rho^2_K + \rho^2_{K_H})/(\epsilon_0 \Omega^2)| << 1$ and $|\delta^2/(\epsilon_0^2 \Omega^2)| << 1$, with $\Gamma^2 = -\Gamma_+ \Gamma_-$. A perturbative calculation of the kinetic contribution to the TAE dispersion relation shows indeed that, for a TAE close to the bottom of the frequency gap ($\Gamma_+ \approx 0$ and $\Gamma_- \approx -2\epsilon_0 \Omega^2$, which is usually the case for the n = 1 and the high-n low shear modes),

$$(124) \quad \Gamma_+ = \beta + \frac{s^2}{2} \left( \frac{1}{\Gamma^2} + \frac{1}{2\epsilon_0^2 \Omega^2} \right) (\rho^2_K + \rho^2_{K_H}) - \frac{s\delta}{4\epsilon_0 \Omega^2} + \frac{\pi^2}{8} \epsilon_0 \Omega^2 \hat{s}^2,$$

with $\Gamma^2 \approx (\frac{1}{2} s^2 \epsilon_0 \Omega^2)^2$. The growth rate $\gamma (\gamma \equiv \text{Im} \omega)$ of the TAE is then given by

$$(125) \quad \frac{\gamma}{\omega_A} \approx \frac{\omega_A}{2\omega(1 + \epsilon_0)q^2} \times \text{Im} \left[ \beta + 4 \left( \frac{8}{\pi^2} \hat{s}^2 \right) \left( \frac{1 + \epsilon_0}{\epsilon_0} \right)^2 (\rho^2_K + \rho^2_{K_H}) - \frac{s\delta}{4\epsilon_0 \Omega^2} \left( \frac{1 + \epsilon_0}{\epsilon_0} \right) \right].$$

Equation (125) includes, in addition to the energetic-particle drive, the TAE damping due to a small kinetic-like perturbation superimposed on the regular MHD wave [30, 60]. From eq. (123), it is possible to see that the finite energetic-particle drift-orbit size (through the terms proportional to $\rho^2_{K_H}$ and $\delta$, respectively) effectively decreases the strength of energetic-particle drive because of resonance detuning.

For sufficiently high energetic-particle pressures, the TAE downward real-frequency shift produced by energetic-particle compressibility causes the TAE mode to be strongly coupled to the lower KTAE branch and damped because of kinetic convection of energy away from the gap region. The criteria given above for exciting the TAE branch may then be violated. In this case, analogously to what happens in the presence of other dissipative effects (e.g., Landau damping), the KTAE branch is instead excited, usually close to the upper shear Alfvén continuum, $\Gamma_- \approx 0$ [60], since it is affected by a fairly small “radiative damping”. In this case, the upper KTAE’s dispersion relation becomes

$$(126) \quad \Gamma_- = \beta + \frac{s\delta}{2\epsilon_0 \Omega^2} \sqrt{\hat{s}^2 + 2\epsilon_0 \Omega^2 (\rho^2_K + \rho^2_{K_H})} - \frac{s\delta}{2\epsilon_0 \Omega^2},$$

or

$$(127) \quad \frac{\gamma}{\omega_A} \approx \frac{\omega_A}{2\omega(1 - \epsilon_0)q^2} \times \text{Im} \left[ \beta + 2\hat{s} \left( \frac{1 - \epsilon_0}{\epsilon_0} \right) \sqrt{\hat{s}^2 + \frac{1}{2} \left( \frac{\epsilon_0}{1 - \epsilon_0} \right) (\rho^2_K + \rho^2_{K_H})} - 2\hat{s}\delta \left( \frac{1 - \epsilon_0}{\epsilon_0} \right) \right].$$

Both equations (125) and (127) depend linearly on $\text{Im} \hat{\beta}$, indicating, thus, that the growth rate of these modes is expected to increase linearly with the mode number for $k_L \rho_{L_H} \ll 1$, as it was anticipated in the present section on the basis of qualitative considerations.

Note that eqs. (126) and (127) refer to the most unstable mode of the upper KTAE branch, i.e., that corresponding to $\mathcal{N} = 0$ in eq. (119). Furthermore, as already pointed out, eqs. (126) and (127) are obtained in the limit $k^2 \rho^2_{L_H} \ll 1$. For mode numbers such that $k^2 \rho^2_{L_H} \simeq 1$, the TAE dispersion relation is given by [41]

$$(128) \quad \mathcal{Y}(\Omega) \equiv \left( -\frac{\Gamma_+}{\Gamma_-} \right)^{1/2} = \delta T_J (\hat{s}, \alpha) + \delta T_K,$$
where $\delta T_f(\dot{s}, \alpha)$ is associated with the thermal plasma (fluid) contribution to variations in the potential energy, while $\delta T_K$ accounts for the analogous (kinetic) contribution due to energetic particle dynamics, and is given by

$$
\delta T_K = \frac{\pi^2}{4|\dot{s}|} \frac{e_H^2}{m_e^2} \frac{q^2 R^2}{k_\alpha^2} \left( \frac{\Omega^2_H Q F_{H0}}{\Xi (1 + \Xi^2)^{3/2}} \sum_{\ell=1,3} \frac{\omega}{\ell^2 v^2_y/(4q^2 R^2_H) - \omega^2} \right),
$$

with $\Xi^2 = (k_\alpha^2/4)(\rho^2_H + \rho^2_{\alpha H}/2)$, with $\rho^2_H = v^2_A/\Omega^2_H$ and $\rho^2_{\alpha H} = q^2 R^2_H \Omega^2_{\alpha H}/(k_\alpha^2 v^2_y)$. Here, recalling that $\omega \equiv v_A/(2qR_H)$ for the present modes, the resonant denominator evidences the fundamental wave-particle resonances at $v = v_A$ and $v = v_A/3$ [20, 23, 27]. Furthermore, previous results on the gap mode growth rates - increasing linearly with the mode number for $k_\alpha^2 \rho^2_{1,2} < 1$ - and the functional dependence of $\delta T_K$ on $\Xi^2 \propto k_\alpha^2 \rho^2_H$ suggests that the most unstable modes are characterized by $k_\alpha^2 \rho^2_{1,2} < 1$ and $k_\alpha^2 \rho^2_{1,2} > 1$, as discussed earlier in this section.

In eq. (128) it is also possible to include the Landau damping contribution due to thermal electron and ion magnetic drifts by simply replacing $[26, 49]$ $\Omega^2 \rightarrow \Omega^2 - \beta_1$ in the expression for $\Upsilon(\Omega)$, where [20]

$$
\beta_1 = \frac{\pi q^2}{B_0^2} \sum_{j=1,3} \sum_{\ell=1,3} m_j \left( \frac{v^2_y}{2} + v^2_{||} \right)^2 \frac{\omega Q F_{j0}}{\ell^2 v^2_y/(4q^2 R^2_H) - \omega^2}.
$$

Equation (128) must satisfy the “causality constraint” Re$\Upsilon(\Omega) > 0$ [41, 62]. In the simple limit $|\dot{s}|, |\alpha| \ll 1$, for which [41] $\delta T_f(\dot{s}, \alpha) = (\dot{s} \pi/4)(1 - \alpha/\alpha_c(\dot{s}))$, the causality constraint is $\alpha < \alpha_c(\dot{s})$ for sufficiently small energetic particle response, i.e., the condition for having a TAE well inside the frequency gap in the continuous spectrum. This is the typical case of a TAE mode driven unstable by an energetic particle population, which may be considered as a perturbation to the thermal plasma. Clearly, this is no longer the case when $\delta T_K \approx \delta T_f(\dot{s}, \alpha)$ in eq. (128). In fact, shear Alfvén wave can still be excited for $\alpha > \alpha_c(\dot{s})$, provided that the energetic particle drive is sufficiently strong to overcome “continuum damping”, $\Upsilon_C(\Omega) \equiv (\Upsilon(\Omega^2))^{1/2}$ [26, 41, 49, 62]. More precisely, close to marginal stability, these modes have $\omega = \omega_r + i\gamma$, with

$$
\text{Re} \delta T_K(\omega_r) + \frac{\dot{s} \pi}{4} \left( 1 - \frac{\alpha}{\alpha_c(\dot{s})} \right) = 0, \\
\gamma \frac{\omega_r}{\omega_r} = -\frac{|\text{Im} \delta T_K(\omega_r) - \Upsilon_C(\omega_r)|}{\omega_r(\partial/\partial \omega_r) \text{Re} \delta T_K(\omega_r)}.
$$

Equation (131) clearly indicates the existence of a threshold in the energetic particle drive, $\propto \text{Im} \delta T_K$, for the excitation of these modes which, for this reason, are called Energetic Particle Modes (EPM’s) [41]. The threshold condition is set by continuum damping, which is proportional to $\Upsilon_C$. It is evident, already from this preliminary discussion, that EPM’s, which will be discussed in detail below, are not eigenmodes of the thermal plasma and that, therefore, their stability properties and non-linear evolution may be quite different from those of TAE and KTAE.

In refs. [26, 49] it has been shown that also the general “even parity” (2) KTAE

*Footnote:* Here, parity is referred to the mode parity with respect to the transformation $(\theta, \theta_k) \rightarrow (-\theta, -\theta_k)$. Incidentally, we note that TAE are characterized only by even parity.
The dispersion relation may be written in the same form of eq. (128), i.e.,

\[ \Upsilon_{\ell,u}(\Omega) = \delta T_K + \delta T_f(\dot{s}, \alpha), \]

where \( \Upsilon_{\ell} \) and \( \Upsilon_{u} \) refer, respectively, to lower and upper KTAE spectra and are given by

\[ \Upsilon_{\ell} = \frac{\Gamma(1/4 + a_{\ell}/2)}{(2\Delta_K)^{1/4} \Gamma(3/4 + a_{\ell}/2)}, \]

\[ \Upsilon_{u} = \frac{(2\Delta_K)^{1/4} \Gamma(3/4 + a_{u}/2)}{\Gamma(1/4 + a_{u}/2)}. \]

Here, \( \Gamma \) is the Euler \( \Gamma \)-function, \( \Delta_K = \delta^2 \rho_K^2 / (\epsilon_0 \Omega^2) \) and

\[ a_{\ell} = \frac{-i(\Gamma_+ - \beta_1)}{\sqrt{2\delta \rho_K / \epsilon_0 \Omega^2}}, \]

\[ a_{u} = -\frac{(\Gamma_+ - \beta_1)}{\sqrt{2\delta \rho_K / \epsilon_0 \Omega^2}}. \]

In the case \( |\delta T_f| \ll 1 \) and with weak energetic particle drive, eq. (132) is easily reduced to [26, 49]

\[ \frac{(\Gamma_+ - \beta_1)}{\epsilon_0 \Omega^2} \approx -i \sqrt{2 \Delta_K} \left\{ \frac{1}{2} + 2N - \frac{(2N + 1)!!}{(2N + 1)} \times \right\}
\]

\[ \times e^{-i\pi/4} \frac{[\delta T_K + \delta T_f]}{(2 \Delta_K)^{1/4} 2^{N-1} N! \sqrt{\pi}} \right\}, \]

for the lower “even parity” KTAE, whereas the upper KTAE branch turns out to be:

\[ \frac{(\Gamma_+ - \beta_1)}{\epsilon_0 \Omega^2} \approx \sqrt{2 \Delta_K} \left\{ \frac{3}{2} + 2N + \frac{(2N + 1)!!}{2^N N!} \times \right\}
\]

\[ \times \frac{(2 \Delta_K)^{1/4}}{\sqrt{\pi}} [\delta T_K + \delta T_f] \right\}. \]

In addition to the “even parity” KTAE branches, it is possible to demonstrate that “odd parity” KTAE’s also exist and are described, in the \( |\dot{s}|, |\alpha| \ll 1 \) limit, by the dispersion relation

\[ \frac{1}{\Upsilon_{\ell,u}(\Omega)} = -\left[ \frac{\delta \Pi}{4} \left( 1 + \frac{\alpha}{\alpha_c} \right) + \delta T_K \right]. \]

Analogously to the “even parity” KTAE branches, when \( |\delta T_f| \ll 1 \) and energetic particle drive is weak, eq. (137) reduces to

\[ \frac{(\Gamma_+ - \beta_1)}{\epsilon_0 \Omega^2} \approx -i \sqrt{2 \Delta_K} \left\{ \frac{3}{2} + 2N - \frac{(2N + 1)!!}{2^N N!} \times \right\}
\]

\[ \times e^{i\pi/4} \frac{(2 \Delta_K)^{1/4}}{\sqrt{\pi}} \left[ \delta T_K + \frac{\delta \Pi}{4} \left( 1 + \frac{\alpha}{\alpha_c} \right) \right] \right\}, \]
for the lower KTAE branch, and to
\[
\frac{(\Gamma_- - \beta_1)}{\epsilon_0 \Omega^2} \simeq \sqrt{2 \Delta_K} \left\{ \frac{1}{2} + 2 N + \frac{(2N + 1)!!}{(2N + 1)} \times \right\}
\]
\[
\times \left[ \frac{\delta T_K + \frac{8\pi}{4} \left( 1 + \frac{\alpha}{\alpha_c} \right)}{(2\Delta_K)^{1/4} 2^{N-1} N \sqrt{\pi}} \right],
\]
(139)

for the upper KTAE branch. To the lowest order, it is readily seen that eqs. (135), (136), (138), and (139) reduce to eqs. (119), as expected.

Equation (137) helps clarifying the reason why a “odd parity” TAE mode cannot exist, since [26, 49] it should fulfill the dispersion relation
\[
\frac{1}{\bar{\Upsilon}(\Omega)} = \left( \frac{\Gamma_- - \beta_1}{(\Gamma_+ - \beta_1)} \right)^{1/2} = - \left[ \frac{8\pi}{4} \left( 1 + \frac{\alpha}{\alpha_c} \right) + \delta T_K \right].
\]
(140)

Equation (140) can never satisfy the causality constraint Re\(\bar{\Upsilon}(\Omega)\) > 0 for sufficiently small energetic particle contribution, i.e., in the typical condition in which a TAE could be excited. On the contrary, eq. (140) may describe the resonant excitation of an “odd parity” EPM, for which an equation similar to eq. (131) can be written above a certain threshold in energetic particle density.

2.5.5. Energetic Particle Modes

That a close relationship exists between TAE, KTAE and EPM can be readily seen from a comparison between eqs. (128) and (140) on one side and eqs. (132) and (137) on the other. In fact, \(\bar{\Upsilon}_{e,\Omega}(\Omega) \rightarrow \bar{\Upsilon}(\Omega)\) as \(|\alpha_{e,\Omega}| \rightarrow \infty\). Thus, when the perturbative assumption that yields eqs. (135), (136), (138), and (139) breaks down, the response of the thermal plasma can be considered “ideal” and all modes (TAE, KTAE and EPM) are described by the same dispersion relations, eqs. (128) and (140), any distinction between them being reduced to differences in the mode growth-rate. In fact, a careful examination of eqs. (135) and (139) indicates that, if KTAE’s are driven strongly enough that the perturbative treatment of energetic particle dynamics fails, the even and odd ideal EPM branches get excited and emerge, respectively, from the closest even lower KTAE and odd upper KTAE [26, 49]. Using the Stirling approximation to factorials, the condition for break down of perturbative analyses reads, respectively,
\[
|\delta T_K| \gtrsim N^{1/2} |\Delta_K|^{1/4} \sim \left| \frac{(\Gamma_+ - \beta_1)}{\epsilon_0 \Omega^2} \right|^{1/2},
\]
(141)

which is equivalent to the threshold condition for EPM excitation, eq. (131). This result demonstrates that the most unstable energetic-particle-driven shear Alfvén gap modes are “ideal modes”. In the absence of TAE \((\alpha > \alpha_c)\) and EPM (i.e., below the threshold condition set by eq. (131)), only moderately unstable upper KTAE can be excited [26, 49], the lower branch being generally more stable [30].

The merging of the odd EPM mode into an odd mode of the upper KTAE branch is shown in figs. 15 and 16. There, for simplicity, only two modes of the upper KTAE spectrum are shown, as obtained from a numerical solution of the dispersion relations eq. (132) (boxes) and eq. (137) (crosses). The energetic particle response, eq. (129), is
computed assuming a slowing-down distribution function, with birth speed $1.3v_A$. The solution of eq. (140) is also reported (circles) to show that, as $-R_0q^2\beta^i_H$ increases, the odd KTAE clearly becomes an unstable EPM, whereas the even KTAE growth rate remains essentially unchanged. The sharp change in the slope of the growth-rate for increasing $-R_0q^2\beta^i_H$ indicates the existence of a threshold for EPM excitations, which may be well identified with that of eq. (131).

![Graph showing growth rate vs. $-R_0q^2\beta^i_H$](image)

**Fig. 15.** - Normalized growth rate, $\text{Im} \left( \Gamma^+ - \beta_1 \right)$, for two modes of the upper KTAE branch with even (boxes) and odd (crosses) parities. Fixed parameters here are: $\bar{s} = 0.5$, $\alpha = 0$, $\Delta\kappa = 0.01$ and $\beta_1 = -0.06$. An odd EPM is also shown for reference (circles).

The resonant excitation of even parity EPM’s is usually related to finite thermal plasma pressure effects. In fact, as discussed above in this section, TAE’ modes tend to merge into the lower Alfvén continuum when $\alpha > \alpha_e(\bar{s})$. Under such circumstances, such modes become eigenmodes of the lower KTAE branch and tend to be stabilized via convective damping. Thus, above the threshold set by eq. (131), the most unstable mode is a low frequency EPM [63]. Figure 17 shows how such a transition from unstable TAE to unstable EPM modes occurs as $\alpha$ is increased above $\alpha_e$. These results are obtained from numerical solutions of eq. (120), using $\epsilon_0 = 0.1$, $\bar{s} = 0.32$, $-R_0q^2\beta^i_H = 0.121$ and $v_H/v_A = 1.0$ as fixed parameters. In this case, the numerical threshold for EPM excitation was found to be $-R_0q^2\beta^i_{He} \simeq 0.05$. Note that the transition from TAE to EPM cannot be considered as a smooth one, since both modes coexist.

2.6. Waves in a torus: numerical results. – In the previous section, the problem of linear stability of shear Alfvén modes has been treated for idealized circular-cross-section plasma equilibria on the basis of asymptotic analyses, which are fully justified in the limit of high toroidal mode number. The interest in moderate, or even low toroidal number modes (which can be of great relevance in the interpretation of present plasma experiments) justifies the numerical approach to the problem. Indeed, in this limit the
Fig. 16. - Normalized real frequency, \( \text{Re}(\Gamma_+ - \beta_1) \) for two modes of the upper KTAE branch. An odd EPM is also shown for reference. Fixed parameters and symbols are the same as in fig. 15.

eigenfunction corresponding to global modes are fairly broad, and thus large portions of the plasma discharge will be displaced from the rest position. On the one hand, this fact makes modes that are eventually driven unstable particularly dangerous, because of their effect on the macroscopic properties of the plasma discharge (particle and energy confinement, etc.). On the other hand, it greatly limits the applicability of the asymptotic analyses which make analytic studies possible.

Further motivation for the numerical approach is given by the need of extending the investigation to non-linear evolution of the Alfven modes, and considering realistic shaped-cross-section experimental devices. Although in this paper we will not address this latter problem, it is instructive, before analyzing non-asymptotic and/or non-linear regimes, to show the main features of the Alfven spectrum in a present-day tokamak configuration, namely the Joint European Torus [2] (JET). The JET plasma equilibrium has a shaped poloidal cross section described by the following equations:

\[
\begin{align*}
R &= R_0 \left[ 1 + \left( a/R_0 \right) \cos(\vartheta + \delta \sin \vartheta) \right], \\
Z &= a \kappa \sin \vartheta, 
\end{align*}
\]

where \( \vartheta \) is the poloidal angle, \( a/R_0 = 0.423 \) is the inverse aspect ratio, \( \kappa = 1.68 \) and \( \delta = 0.3 \) are the ellipticity and the triangularity of the plasma boundary, respectively. Equilibria with shaped-plasma cross sections are generally preferred to circular ones because of their improved performance in terms of thermonuclear plasma figures of merit. In fig. 18, the poloidal cross section of the JET tokamak is shown, for an equilibrium obtained with the CHEASE code (see appendix sect. 5'1) assuming a volume average \( \beta \approx 3\% \) and the safety factor \( q \) ranging from \( q(0) \approx 1 \) to \( q(a) \approx 6 \). It is to be noted that
Fig. 17. – Real frequency and growth rates for the two most unstable modes obtained from a numerical solution of eq. (120). The TAE mode is indicated by open circles and the EPM mode by triangles. The growth rates are maximized with respect to $k_\perp \rho_{LH}$. The corresponding peaks of the $qR_0 k_\parallel$ spectrum are also shown.
Fig. 18. – Poloidal cross section for a typical JET equilibrium.
non-circular shapes of the internal magnetic surfaces could be developed even in plasma equilibria with circular boundary because of the natural outward Shafranov shift of the magnetic axis. Besides the gaps in the continuum spectrum induced by toricity, higher order gaps appear, in general configurations, because of coupling of poloidal harmonics \( m \) with \( m \pm 2, m \pm 3, \ldots \), being related, e.g., to ellipticity, triangularity, etc. The continuous Alfvén spectrum for the toroidal mode number \( n = 1 \) is shown in fig. 19, as obtained from the code MARS assuming an ideal plasma with \( \Gamma = 0 \) (that is, excluding the slow magnetosonic wave continuum) and a plasma density \( \rho_0(s) = \rho_0(0)(1 - s^2) \). Gaps centered around \( \omega / \omega_A \approx 0.4 \) (the toricity induced gap) and \( \omega / \omega_A \approx 0.9 \) (the ellipticity induced gap) appear in the Alfvén continuum. Also the dominant poloidal components in the Alfvén continua are marked on the curves. Also, the global gap modes are shown.

![Alfvén frequency spectrum](image)

Fig. 19. – Alfvén frequency spectrum (for toroidal mode number \( n = 1 \)) for the JET equilibrium shown in fig. 18. The dominant poloidal components in the Alfvén continua are marked on the curves. Also, the global gap modes are shown.

2.6.1. Global toroidal modes

We start the investigation of the non-asymptotic low-\( n \) limit considering a circular-cross-section equilibrium, with safety factor \( q \) ranging from \( q \approx 1.1 \) to \( q \approx 1.8 \), inverse aspect ratio \( \epsilon = 0.1 \), flat radial density profile, and zero plasma pressure. Consistently with the choice of a small value of \( \epsilon \), we will take into account only the coupling between adjacent poloidal components. The eigenvalue version of the MARS code will be used (see appendix sect. 5.2), generalized to a complex resistivity to represent in a simple way
the kinetic effects associated to finite ion Larmor radius.

In fig. 20 the Alfvén continuous spectrum for the \( n = 1 \) mode is shown versus the radial-like variable \( s \), considering only the poloidal mode components \( m = 1, 2 \).

![Alfvén continuous spectrum](image)

Fig. 20. – Upper and lower branches of the Alfvén frequency spectrum for the circular cross section case, \( n = 1, m = 1, 2 \) (solid lines). Also the cylindrical \( m = 1 \) and \( m = 2 \) continuous spectra are shown (dashed lines).

The cylindrical Alfvén continuous spectra are also shown (dashed lines). At the radial location \( s_0 \) where the safety factor is \( q(s_0) = (2m + 1)/(2n) \), and where the two cylindrical continua are degenerate, a “gap” appears, the width of which is ordered with the local inverse aspect ratio \( \epsilon_s = r(s_0)/R_0 \). In fig. 21, the radial velocity eigenfunction is shown for the TAE mode, which lies within the gap at \( \omega/\omega_A \approx 0.316 \). It is a global mode, in the sense that the eigenfunction is radially extended over a large portion of the plasma cross section. Such eigenfunction exhibits the larger variation around the gap position. This mode is a purely oscillating mode (marginally stable mode) in ideal MHD. Adding non-ideal terms to the MHD equations (as, e.g., resistivity, finite ion Larmor radius, parallel electron dynamics or kinetic effects associated with electron Landau damping) slightly modifies the nature of the TAE, the main difference being the appearance of a complex part of the eigenfrequency, but it gives rise, as discussed in sect. 2.5, to the KTAE’s or their purely resistive version, the RPSAE’s.

In figs. 22 and 23 the real frequency and the damping rate of the TAE and of the less stable upper and lower branches of the RPSAE’s (corresponding to \( N = 0 \) in eqs. (119)) are shown versus the inverse of the Lundquist number, \( S^{-1} \equiv \eta v_c^2 R_0/(4\pi n^2 v_A) \). The results for the RPSAE’s slightly depart from the perfectly-symmetric theoretical predictions discussed after eqs. (119) because of the weak radial dependence of the equilibrium quantities. Their damping rate scales as the square root of the resistivity \( \gamma \propto \eta^{1/2} \), whereas the TAE damping rate is found [30, 64] to scale as \( \gamma \propto \eta^{1/3} \) in the limit of large
resistivity, \( \eta n^2 \rho(s_0)^3 [a/r(s_0)]^2 \ll 2\epsilon s_0 \), and as \( \gamma \propto \eta/\epsilon s_0 \) in the limit of small resistivity, \( \eta n^2 \rho(s_0)^3 [a/r(s_0)]^2 \gg 2\epsilon s_0 \).

The damping is due to the coupling with the KAW’s, and is associated to the propagating component of such waves, which carry energy away from the gap region \( \text{(radiative damping)} \). In the case of the TAE (whose frequency is properly contained within the forbidden gap), the KAW’s can only be excited by \textit{tunneling} [32], and thus the damping is typically smaller than for RPSAE.

The inclusion of kinetic effects results in increasing the damping of the lower branch, whereas the upper branch tends toward the marginal stability. This can be seen from fig. 24, which has been obtained from MARS simulations adding an imaginary part to the resistivity to model kinetic terms.

In fig. 25 and fig. 26, the radial velocity eigenfunctions are shown for an upper \( (\omega/\omega_A = 0.416) \) and lower \( (\omega/\omega_A = 0.293) \) KTAE, respectively. A complex resistivity corresponding to \( S^{-1} = (1 - 2i) \times 10^{-6} \) has been assumed. The propagating character of the eigenfunction is clearly visible in the gap internal region \( (0.7 < s < 0.9) \) for the upper branch and in the external region \( (s < 0.7 \text{ and } s > 0.8) \) for the lower branch. A typical TAE \( (\omega/\omega_A = 0.318) \) is shown, in the presence of kinetic effects, in fig. 27 and is to be compared with the ideal TAE of fig. 21: only small oscillations of the eigenfunction inside the gap region are present, which are typical of the (weak) coupling to KAW’s.

Some general considerations can be done on the parity of the eigenfunctions of the global mode considered above. Following ref. [40] an expression for the radial derivative of the Fourier components \( \delta \tilde{\phi}_{1,1} \) and \( \delta \tilde{\phi}_{2,1} \) of the scalar potential can be obtained from eq. (82):

\[
\frac{d \delta \tilde{\phi}_{2,1}}{d \delta \tilde{\phi}_{1,1}} / dr \propto \frac{\omega^2 / \omega_A^2 - k_{1,1}^2 R_0^2}{\omega^2 / \omega_A^2} \approx \frac{\omega^2 / \omega_A^2 - 1/9}{\omega^2 / \omega_A^2}.
\]
Fig. 22. – Real frequency of TAE, upper and lower RPSAE versus resistivity.

Fig. 23. – Damping rate of TAE, upper and lower RPSAE versus resistivity.
Fig. 24. – Complex frequency plane for gap modes. The open circle refers to the ideal TAE. Full circles refer to the resistive TAE and RPSAE ($S^{-1} = 10^{-5}$). Open and full squares refer to TAE and KTAE ($S^{-1} = (1 - i) \times 10^{-5}$ and $S^{-1} = (1 - 2i) \times 10^{-5}$, respectively). Also the upper and lower Alfvén continua are shown.

Fig. 25. – Radial profile of the poloidal harmonics of $v^*$ for the upper KTAE. Real (full line) and imaginary (dashed line) part of the eigenfunctions are shown.
Fig. 26. – Radial profile of the poloidal harmonics of $v'$ for the lower KTAE.

Fig. 27. – Radial profile of the poloidal harmonics of $v'$ for the kinetic TAE.
The upper branch of the KTAE is characterized by \( \omega^2/\omega_A^2 > 1/9 \), and thus the two different Fourier components are out of phase as can be observed in fig. 25. On the contrary, the lower KTAE and, for typical equilibria, also the TAE are characterized by \( \omega^2/\omega_A^2 < 1/9 \), and their Fourier components are mutually in phase (figs. 21 and 26).

### 2.6.2. Continuum damping

In the present section we study the continuum damping of a TAE mode, considering an equilibrium similar to the one used in the case shown in fig. 20, but imposing \( q(0) = 1.4 \), thus shifting the localization of the gap toward the center of the plasma column. Moreover, a radial density profile \( \rho_0(s) = \rho_0(0)(1 - s^4)^c_\rho \) is used, with \( c_\rho > 0 \), causing the Alfvén continuum frequency to increase at the plasma edge. This choice allows us to better separate the gap region (where the mode is localized) from the plasma boundary, where the interaction with the continuum takes place. Considering, as before, the \((1,1)\) and \((2,1)\) harmonics, the gap is localized around \( s = 0.5 \), and the TAE resonantly excites the continuum for \( c_\rho > 0.5 \). The damping rate can be obtained from the MARS code taking the limit of vanishing resistivity for a resistive TAE (see, e.g., ref. [65]) shown, at different values of \( L_A \) (cf. eq. (110)), in fig. 28. Note that the scaling observed in fig. 28

![Graph](image)

**Fig. 28.** Continuum damping rate versus \(|L_A|/a\) (with \(|L_A|\) being the scale-length of \(g_0 \rho_0^2\)).

is apparently opposite with respect to that \( \approx |L_A|^{-3/2} \), apparently given by eq. (110) and usually quoted in literature for the high-\( n \) case. In fact, in the present low-\( n \) case, changing the \( c_\rho \) coefficient also modifies the tunneling factors in the same equation. This effect cannot be considered as a minor correction to the power-scaling result, due to the exponential dependence of such factors.

This shows that, when TAE has significant amplitude at the radial locations where the Alfvén continuum is resonantly excited, strong coupling to the continuous spectrum and large mode damping due to phase mixing can occur. This is true, a fortiori, in those
situations in which the interaction takes place in a region where the mode is not in cut-off. One of such situations occurs when the plasma $\beta$ is raised above a critical threshold [38]. In fact, finite $\beta$ causes the local TAE frequency to shift downwards, eventually touching the lower boundary of the frequency gap, where the eigenfunction is close to its maximum amplitude. It has to be noted that in the case of TAE modes merging into the lower continuum, analytic treatments, which are usually based on perturbation theories, are no longer valid and fully non-perturbative treatments, as those obtainable by numerical simulations, are needed to properly describe the mode dynamics.

2.6.3. Kinetic effects

In sect. 2.5 it has been pointed out that, depending on the relative importance of toroidal equilibrium effects vs. kinetic dissipation terms, TAE’s may exist in their “regular” MHD or in their kinetic modified version, the KTAE. Aim of the present section is to compare those analytical estimates with the results yielded by linear simulation performed by the linear hybrid MHD-gyrokinetic numerical code (see appendix sect. 5.5). Such simulations are performed by using the $\delta F$ algorithm (i.e., numerically evolving only the perturbed part of the energetic-particle distribution function) and retaining only the unperturbed terms when advancing the phase-space particle coordinates (the gyrocenter position $\mathbf{\bar{R}}$, the magnetic moment $\mathbf{\bar{M}}$ and the parallel velocity $\mathbf{\bar{U}}$) according to eqs. (192) (see appendix sect. 5.5); the linearly perturbed terms are instead included in the evolution of the weight factors, eq. (207). Furthermore, the mirroring term, i.e., the term proportional to $\mathbf{b} \cdot \nabla \ln B_0$ in $d\mathbf{U}/dt$, responsible for particle trapping is neglected, as our analysis is focused, for simplicity, on the effects of circulating particles.

We consider an equilibrium magnetic field characterized by $\psi^{eq}(r, \phi) = \psi_0^{eq}(r) + \psi_1^{eq}(r) \cos \phi$, with $q$-profile approximately given by $q(r) \approx q(0) + [q(a) - q(0)] r^2/a^2$, with $q(0) = 1.1$ and $q(a) = 1.9$. Perturbations with two coupled harmonics, $(1,1)$ and $(2,1)$, are taken into account for $\psi$ and $\phi$. With the equilibrium considered here, the frequency spectrum for the shear Alfvén modes shows a single-gap structure with the gap localized at $\tilde{r} \equiv r/a \approx 0.7$. A TAE-like radial mode structure is chosen for the Fourier components of the initial electromagnetic perturbation (the scalar potential is shown in fig. 29), resulting from a preliminary simulation performed with a model driving term. Note that the two poloidal harmonics of the scalar potential are mutually in phase, as expected on the basis of eq. (143).

The time evolution of the two Fourier components of the total (magnetic plus kinetic) fluctuating energy for the hybrid simulation with inverse aspect ratio $a/R_0 = 0.25$, $v_H/v_A = 1$, $\beta_H(0) = 0.04$ and $m_H/m_i = 2$ is shown in fig. 30. The other parameters are the following (see appendix sect. 5.5): $\rho_n H/a = 0.01$, $a_n = 2$, $a^2/L_n^2 = 2$, $S^{-1} = 10^{-5}$. In spite of the strong energetic-particle drive, the initial configuration incoherently decays, indicating that, due to the strong coupling with KAW’s, the TAE is damped, contrary to what could be inferred from a perturbative treatment of the energetic-particle dynamics.

Later on in the simulation, the fluctuating energy starts to grow, because an unstable KTAE emerges. This can be seen from fig. 31, which shows the perturbed-field profiles at $t = 576 \omega_A^{-1}$. The KTAE upper-branch symmetry $\partial_r \delta \phi_{2,1} / \partial_r \delta \phi_{1,1} < 0$ is evident.

This result [61] can be understood from fig. 32, where the analytical estimates for the growth rate of the TAE, eq. (125), and that of the KTAE, eq. (127), are plotted versus the on-axis $\beta_H$ value. The value corresponding to the simulation is also shown. For this value the KTAE is predicted to be more unstable than TAE. Without finite drift-orbit size effects ($\beta_H = \beta_{KH}^2 = \delta = 0$), eqs. (125) and (127) would have indeed
predicted the TAE mode to be more unstable than the upper KTAE branch, showing
the essential role played by these effects in bringing the TAE mode so close to the lower
shear Alfvén continuous spectrum that it is completely stabilized by the coupling to the
lower KTAE branch, as discussed in sect. 2.5.

The growth rate of the KTAE, obtained from numerical simulations at a fixed value
of $\beta_H(0) = 0.04$ and different values of $v_H/v_A$, is shown in fig. 33 along with the analytical
eigenvalue (dashed line) obtained from eq. (127). The main qualitative features are well
reproduced: the maximum growth rate at $v_H \approx v_A$, corresponding to the resonant
center of the driving mechanism, and the effect of the secondary resonance at $v_H \approx 3v_A/4$. The quantitative agreement, however, appears to be quite poor. This can be
attributed to the approximations adopted in calculating the expressions for $\tilde{\beta}$, $\tilde{\delta}$ and $\tilde{\rho}_{KH}$
(eqs. (123)). In particular, it has been assumed that the energetic-particle contribution
to the driving and damping terms comes essentially from the gap region, where the field
amplitude is maximum. This is justified with regard to the finite Larmor radius terms $\tilde{\rho}$
and $\tilde{\rho}_{KH}$, whilst it is not a good assumption for the driving term, $\tilde{\gamma}_H \equiv \text{Im} [\tilde{\beta} \omega_A/(2 \omega(1-
\epsilon_0)q^2)]$, which receives significant contributions from the whole radial extension of the
eigenmode for realistic (not too large) values of the aspect ratio of the torus. A more
accurate estimate of $\tilde{\gamma}_H$ can be obtained in terms of the energetic-particle contribution
to the potential energy $\delta W_K$ and of the plasma kinetic energy $K_M$, as [23]

$$
\tilde{\gamma}_H \simeq \text{Im} \lim_{\text{Im } \omega \to 0} \frac{\delta W_K}{K_M} \simeq \\
- \frac{1}{2 \rho_{LH}} \frac{v_H}{v_A} \frac{m_H}{m_i} \frac{n_{H0}}{n_0} \text{Re} \int d^3\tilde{\phi} \delta \tilde{P} \hat{B} \times \nabla R \cdot \nabla \delta \tilde{\phi}^* \\
\int d^3\tilde{\phi} |\nabla \cdot \tilde{\phi}|^2.
$$

(144)

The value of $\tilde{\gamma}_H$ can then be computed by using the following approximate expressions
Fig. 30. - Time evolution of the Fourier components $m = 1, 2$ and $n = 1$ of the total (magnetic plus kinetic) fluctuating energy for the hybrid simulation with $v_H/v_A = 1$, $\beta_H(0) = 0.04$, $m_H/m_i = 2$, $a/R_0 = 0.25$, $\rho_{LH}/\alpha = 0.01$, $\alpha_n = 2$, $a^2/L_\alpha^2 = 2$, $S^{-1} = 10^{-5}$. The initial TAE-like configuration incoherently decays, because of the strong coupling with KAW's. Later on, the fluctuating energy starts to grow due to the formation of an unstable KTAE.
Fig. 31. - Radial profiles of the fluctuating-potential Fourier components at $t = 576 \omega_A^{-1}$. The KTAE upper-branch symmetry, $\partial_r \delta \phi_{2,1} / \partial_r \delta \phi_{1,1} < 0$, is consistent with the occurrence of a peak in the frequency spectrum close to the upper boundary of the gap.

Fig. 32. - Analytical estimates for the growth rates of the TAE (dashed line) and KTAE (solid line) versus $\beta_H(0)$. The value chosen for the simulation is indicated by an arrow. For this value the KTAE is predicted to be more unstable than the TAE.
Fig. 33. – Growth rate of the KTAE mode versus \( v_H/v_A \), at a fixed value of \( \beta_H(0) = 0.04 \). Open squares refer to the hybrid-code simulations. The dashed line corresponds to eq. (127), in which only the local contribution to the drive from the gap region is retained. The solid line was obtained by replacing, in eq. (127), the local drive with the expression given by eq. (144), computed in terms of the scalar-potential profiles obtained from the hybrid simulations with \( v_H/v_A = 1 \).

for the perturbed pressure \( \delta \hat{P} [61] \):

\[
\delta \hat{P}_{m,n}(r, t) \simeq -\frac{2m}{n_{H0}} \frac{R_0}{a} \frac{v_H}{v_A} \frac{\rho_LH}{a} \frac{\omega_A}{\omega} \left( \frac{\hat{\delta} \phi_{m,n}}{r} \right) \left( \frac{\partial \hat{\delta} n_H}{\partial r} \right)
\]

\[
\delta \hat{P}_{m\pm,1,n}(r, t) \simeq \pm \frac{m}{n_{H0}} \frac{R_0}{a} \frac{v_H^2}{v_A} \frac{\rho_LH^2}{a^2} \left( \frac{\omega_A}{\omega} \right)^2 \left( \frac{\partial}{\partial r} + \frac{m}{r} \right) \left( \frac{\hat{\delta} \phi_{m,n}}{r} \right) \left( \frac{\partial \hat{\delta} n_H}{\partial r} \right) \times
\]

\[
z_{\pm} \left\{ 3z_{\pm} + 2z_{\pm}^3 + \left[ 1 + 2z_{\pm}^2 \left( 1 + z_{\pm}^2 \right) \right] Z(z_{\pm}) \right\},
\]

(145)

and using the scalar-potential profiles obtained from the hybrid simulations with \( v_H/v_A = 1 \). In eqs. (144) and (145), we have defined

\[
\delta \hat{P} \equiv \delta \hat{P}_\parallel + \delta \hat{P}_\perp,
\]

\[
\delta \hat{P}_\parallel = \left( \frac{2\pi}{n_{H0} T_H} \right) \int dM d\Omega_H \Omega_H^2 \delta \hat{F}_H,
\]

and

\[
\delta \hat{P}_\perp = \left( \frac{2\pi}{n_{H0} T_H} \right) \int dM d\Omega_H \frac{\Omega_H^2}{m_H} \delta \hat{F}_H,
\]

with \( \delta \hat{F}_H \) being the perturbed energetic-particle distribution function in the gyrocenter coordinate system. Moreover, \( \delta \dot{\phi} \equiv e_H \delta \phi / T_H \), \( z_{\pm} \equiv \omega v_A / \left[ \sqrt{2} \omega_A v_H (m \pm 1) / q - n \right] \), and \( Z \) is the plasma dispersion function \([66]\).

The solid line in fig. 33 is the result of eq. (125), in which \( \text{Im} \hat{\beta} \) is substituted with \( 2(\omega/\omega_A)(1 - c_0)q^2 \tilde{\gamma}_H \), and \( \tilde{\gamma}_H \) is obtained from eq. (144). The agreement with the simulation findings is now excellent, showing the relevance that details neglected
in the asymptotic treatment can assume in realistic situations, and, in particular the importance of the broad character of the eigenfunction for low-$n$ modes.

Let us now look at the effect of increasing the energetic-particle drive. In fig. 34(a) the growth rate of the most unstable modes and the corresponding real frequency are plotted at different values of $\beta_H(0)$, for $a/R_0 = 0.1$ and the other parameters as in fig. 30. Figure 34(b) shows the gap structure corresponding to the equilibrium and the mode numbers considered here. Two different regimes can be recognized. At low values of $\beta_H$, the KTAE is the most unstable mode. The real frequency $\omega_r$ of the mode comes out to be close to the upper boundary of the gap and the mode structure — analogous to that shown in fig. 31 — exhibits its sharpest variation in correspondence of the gap region. The growth rate is weakly dependent on $\beta_H$, and satisfies the condition $\gamma \ll \omega_r$. Above a certain threshold in $\beta_H$ ($\beta_H(0) \approx 0.024$ in this case) a new mode appears, with growth rate sharply increasing with $\beta_H$. Its real frequency becomes very small (so that $\gamma \approx \omega_r$) and falls inside the lower continuum. The mode structure, reported in fig. 35 for $\beta_H(0) = 0.03$, shows the $m = 1$ and $m = 2$ poloidal components localized at $r \approx 0.3a$ and $r \approx 0.8a$, respectively. Such localization appears to be mainly determined by the boundary conditions and the resonant excitation of the shear Alfvén continuous spectrum at the mode frequency, rather than by the gap position. Differently from the odd-symmetry KTAE case and consistently with the opposite localization of the real frequency with respect to the centre of the gap, the two poloidal components show even symmetry. All these features allows us to identify this mode as the EPM [26, 41].

It is worth to note that, while the TAE (or the KTAE) exists as a MHD (or kinetic) global mode and is affected by energetic-particle dynamics in a perturbative way (both mode structure and real frequency are substantially determined by the MHD terms in eqs. (180) and (183)), the EPM exists only because of the energetic-particle contribution. Not only its growth rate, but also its localization and real frequency are, in fact,
Fig. 35. – Radial profiles of the scalar-potential components for a simulation with $\beta_R(0) = 0.03$.

The two components are peaked quite far from the gap position and show even symmetry. Their localization is determined by the boundary conditions and the resonant excitation of the shear Alfvén spectrum at the mode frequency. This mode can be identified as the Energetic Particle Mode.

determined by the competition between energetic-particle resonant drive and continuum damping. In this respect, the presence of a frequency gap, needed for the formation of MHD/kinetic global modes, is not directly related to their existence.

These features of the EPM strongly limit the validity of any perturbative treatment of the linear stability of shear Alfvén modes. We can appreciate this fact by comparing the results obtained by the just described self-consistent simulations with those of simplified “perturbative” ones. These latter are performed by approximating the evolution of the perturbed fields in the following way: both the real frequency and the radial profile of the mode are fixed, while the growth rate is computed, at each step, by retaining the energetic-particle contribution on the basis of an approximate expression like that of eq. (144). The damping mechanisms are surrogated by a fixed background dissipation rate $\gamma_D$. This approach bypasses the MHD field solver and clearly corresponds to neglecting the effects of the particles on the mode structure and on its frequency, and closely resembles that of refs. [67, 68, 69], with a little difference: the real frequency shift due to energetic particles, neglected here, is evaluated there on the basis of a perturbative expression analogous to that used to compute the growth rate.

The comparison between the above self-consistent results and the findings of perturbative simulations is presented in fig. 36. The radial profiles adopted, for the fluctuating potentials, in such perturbative simulations correspond to a $n = 1$ KTAE-like eigenmode. A fixed value $\omega_R = 0.33 \omega_A$ has been assumed for the real frequency. While the two treatments show fair agreement in the KTAE regime, the EPM destabilization cannot be reproduced in the perturbative framework.

The results on the transition from gap modes (TAE’s and KTAE’s) to continuum
Fig. 36. - Growth rate versus $\beta_H(0)$ for self-consistent (circles) and perturbative (boxes) linear simulations in the $n = 1$ case. A fixed real frequency $\omega_r = 0.33 \omega_A$ has been assumed for perturbative simulations. Perturbative results agree quite well with the self-consistent ones in the KTAE regime, but do not exhibit the transition to the EPM unstable regime.
modes (EPM’s) seem consistent with those reported, for the high-\( n \) limit, in sect. 25.5. Note that the present investigation, as that of ref. [63], suggests that such transition cannot be described as a smooth one. The coexistence of the two distinct modes, with sharply different real frequencies, can indeed be made evident by looking at the transition regime, where the respective growth rates are of the same order. Figure 37 shows the time evolution (a) of the \( m = 2 \) component of the fluctuating scalar potential and the corresponding frequency spectrum (b) at \( r = 0.5 \alpha \), for \( \beta_H(0) = 0.024 \). Both the KTAE and the EPM are visible, with real frequencies \( \omega_r \approx 0.56 \omega_A \) and \( \omega_r \approx 0.14 \omega_A \) respectively.

Fig. 37. – Time evolution (a) and corresponding frequency spectrum (b) of the \( m = 2 \) fluctuating scalar potential at \( r = 0.5 \alpha \), for a simulation with \( \beta_H \) in the transition regime from KTAE to EPM (\( \beta_H(0) = 0.024 \)). The coexistence of unstable EPM and KTAE, with real frequencies \( \omega_r \approx 0.14 \omega_A \) and \( \omega_r \approx 0.56 \omega_A \) respectively, can be observed.

Similar results are found in the case of higher – although still moderate – toroidal mode number \( n \). In fig. 38(a) real frequencies and growth rates are reported versus \( \beta_H(0) \) for the case of modes with \( n = 4 \) and \( m \) ranging from 4 to 8 (only the poloidal components that yield, for the considered \( q \) profile, frequency-gap positions properly contained inside the plasma volume have been retained). The corresponding shear Alfvén continuous spectrum is shown in fig. 38(b). For \( \beta_H(0) \gtrsim 0.02 \), the real frequency is so low, compared with the growth rate, that it cannot be determined on the basis of the numerical-simulation results. The destabilization of EPM’s is observed, in this case, at much lower values of \( \beta_H \) than those observed in the \( n = 1 \) case. This can also be seen from fig. 39, where the \( n \) dependence of the threshold \( \beta_H(0) \) is shown. In order to qualitatively explain this dependence, we assume that the continuum damping of the mode does not depend on \( n \), which is certainly true in the high-\(n\) limit (cf. eq. (131)). Moreover, it has been shown that the energetic-particle drive, close to the threshold, is proportional to \( \beta_H \) [41], and that it grows linearly with \( n \) for low \( n \) values, while it decreases as \( n^{-3} \) in the large-\( n \) limit [25, 26, 41, 62, 70]. By balancing damping and drive terms, \( \beta_{Hd} \) can then be predicted to decrease as \( n^{-1} \) for low-\( n \) modes and to increase as \( n^3 \) at large \( n \), so that it is expected to reach a minimum at some intermediate value of \( n \). Simulation results show that such a minimum is not yet reached up to \( n = 8 \), so that higher-\( n \) modes can be expected to have a even lower excitation threshold and to play a role in reactor-relevant tokamak plasmas.

3. – Non-linear Theory
Fig. 38. – Real frequency (boxes) and growth rate (circles) of the most unstable mode at different $\beta_H(0)$ values for linear simulations of modes with $n = 4$ and $m$ ranging from 4 to 8 (a), and the corresponding shear Alfvén continuous-spectrum (b). The real frequency for $\beta_H(0) = 0.02$ is too low, compared with the growth rate, to be appreciated from the numerical-simulation results. The threshold for the EPM destabilization is much lower than in the $n = 1$ case.

Fig. 39. – Threshold $\beta_H(0)$ value for the EPM destabilization at different values of the toroidal mode number $n$. The results corresponding to $n = 8$ should be considered just as indicative ones, because of the reduced velocity-space resolution achieved in the corresponding high-spatial-resolution simulations.
3.1. Fluid non-linearities. – In the following, we discuss the non-linear dynamics of the shear Alfvén spectrum in a toroidal plasma. More specifically, we analyze the saturation of TAE modes via mode-mode couplings, assuming that slow beat wave fluctuations are characterized by a phase velocity much smaller than the sound speed. Thus, the issue of resonant interactions with ions of the thermal plasma of sound-wave-like non-linear fluctuations, produced by ponderomotive forces associated with TAE’s, is neglected here. This topic is discussed in great detail in ref. [71], where TAE saturation via frequency cascading, due to non-linear ion Landau damping of low-frequency waves excited by the beating of two TAE’s, is analyzed.

Following sect. 2.5, the non-linear equations for TAE’s will be derived from the quasi-neutrality condition \( \nabla \cdot \mathbf{J} = 0 \); i.e., eq. (62). As shown there, the perpendicular current perturbation is obtained from the perpendicular force balance, where, now, also non-linear terms are kept; i.e.,

\[
\delta \mathbf{J}_\perp \left( 1 + \frac{\delta B_\parallel}{B_0} \right) = \frac{c}{B_0^2} \mathbf{B}_0 \times (\varrho_0 + \delta \varrho) \frac{\partial}{\partial t} \delta \mathbf{v}_\perp + \frac{c}{B_0} \mathbf{B}_0 \times \nabla \delta P + \\
(J_0 \parallel + \delta J_\parallel) \frac{\delta \mathbf{B}_\perp}{B_0} - \frac{\delta B_\parallel}{B_0} \frac{c}{B_0^2} \mathbf{B}_0 \times \nabla P_0 + \\
\left( \frac{c}{B_0^2} \mathbf{B}_0 \times \varrho_0 (\delta \mathbf{v}_\perp \cdot \nabla) \delta \mathbf{v}_\perp \right).
\]

Using the expression for \( \delta J_\parallel \) as given by the parallel Ampère’s law and substituting eq. (146) back into the quasi-neutrality condition, we obtain the following non-linear vorticity equation

\[
\frac{\partial}{\partial t} \left\{ \left( \mathbf{B}_0 + \delta \mathbf{B}_\perp \right) \cdot \nabla \left[ \frac{1}{B_0} \nabla^2 \left( \frac{1}{c} \delta A_\parallel \right) \right] + \nabla \cdot \left\{ \left( \frac{1}{v^2} \right) \frac{\partial}{\partial t} \left[ - \left( 1 + \frac{\varrho_0}{\varrho_0} \right) \frac{\partial}{\partial t} \right] \delta \mathbf{v}_\perp \cdot \nabla \delta \phi \right\} + 8 \pi \kappa \times \frac{\mathbf{B}_0}{B_0^2} \nabla \left[ \frac{\delta \mathbf{v}_\perp}{c} \cdot \nabla \left( P_0 + \delta P \right) \right] \right\} = 0,
\]

where \( \delta \mathbf{v}_\perp = (c/B) \mathbf{b} \times \nabla_\perp \delta \phi \). Here, all corrections of order \( O(\epsilon/n) \) or higher have been neglected, and only the leading order non-linear contributions have been retained. Note that the parallel vector potential fluctuation, \( \delta A_\parallel \), is related with \( \delta \psi \), the fluctuation of the poloidal magnetic flux function \( \psi \), as \( \delta A_\parallel = (R/R_0) \delta \psi \). In deriving eq. (147), a high-\( n \) limit has been assumed. Moreover, the equilibrium terms have been separated from fluctuating ones; in this way, we may directly identify the physical processes involved in each of the considered non-linear processes.

Equation (147) is closed by the non-linear parallel Ohm’s law

\[
- \left( \frac{1}{c} \right) \frac{\partial}{\partial t} \delta A_\parallel = \mathbf{b} \cdot \nabla \delta \phi + \frac{\delta \mathbf{v}_\perp}{c} \cdot \nabla \delta A_\parallel.
\]

The non-linear dynamics, described by the coupled Equations (147) and (148), can be interpreted as follows. The term proportional to \( \delta \mathbf{B}_\perp \cdot \nabla_\perp \) accounts for non-linear magnetic field line bending, whereas those proportional to \( \delta \mathbf{v}_\perp \cdot \nabla_\perp \) relate to non-linear \( \mathbf{E} \times \mathbf{B} \) flows. Finally, the terms proportional to \( \delta \varrho \) and to \( \delta P \) account for non-linear fluctuations due to the ponderomotive force \( \left[ \mathbf{B}_0/(cB_0) \right] \cdot (\delta \mathbf{J} \times \delta \mathbf{B}^*) \); in fact, it can be shown that only the slow non-linear fluctuations \( (\delta \varrho_0 / \varrho_0) \) and \( (\delta P_0 / P_0) \) play a significant role for \( \beta < \epsilon/n \), which we assume to be the case in the following.
Our purpose, here, is not to report on the details of the work published elsewhere [44, 71, 72, 73, 74, 75, 76], but to give a discussion of the dynamic properties of TAE’s which determine its non-linear saturation via mode-mode coupling at levels comparable with, or even lower than, that due to non-linear weakening of the energetic particle drive [43], which yields saturated amplitudes \( \delta B_r/B_0 \approx (\gamma/\omega_d)^2 \) (cf. also sect. 3’4). However, it is fundamental to note that, in both cases, the proposed picture of non-linear TAE saturation can be considered correct only below the threshold for stochastic particle orbits.

The fact that mode-mode couplings can be very efficient in determining TAE’s non-linear saturation can be explained recalling that TAE modes exist in a narrow (toroidicity induced) frequency gap in the shear Alfvén continuous spectrum of approximate width \( \epsilon_0 \omega_A \) (cf. eq. (87)). Therefore, the non-linear terms in the coupled equations (147) and (148) can alter the mode dynamics and eventually lead to saturation only when they introduce a time scale shorter than \( (\epsilon_0 \omega_A)^{-1} \), or, more precisely, shorter than the distance (in frequency), \( \Delta_\omega \), of the linear eigenmode from the accumulation point of the shear Alfvén continuous spectrum (in certain conditions, \( |\Delta_\omega| \ll |\epsilon_0 \omega_A| \)).

Previous analyses [44, 72, 73], which have been recently confirmed [76], have considered the effect of non-linear magnetic field line bending and of convective non-linearities associated with the non-linear \( \mathbf{E} \times \mathbf{B} \) drift, showing that TAE saturation may occur, when the non-linear frequency shift becomes comparable with \( \Delta_\omega \), because of mode conversion to damped short-wavelength modes. Meanwhile, the possibility of TAE saturation as a consequence of non-linear \( \mathbf{E} \times \mathbf{B}^* \) sheared flows, when the induced decorrelation rate overcomes \( \Delta_\omega \), was demonstrated in ref. [74] for the low-\( n \) case and in ref. [75] for the high-\( n \) one. Finally, in ref. [75], it was demonstrated that TAE saturation may take place via non-linear density modulations (the term \( \propto (\delta P_\rho/P_0) \) in eq. (147)) and non-linear pressure cavitations (the term \( \propto (\delta P_\rho/P_0) \) in eq. (147)) when the non-linear frequency shift becomes comparable with \( \Delta_\omega \).

In all these cases, the non-linear saturated phase is characterized by fine scale perturbations to the radial mode structures. To see this effect more clearly, in what follows we will further delineate the solutions of eqs. (147) and (148) in two particular cases, i.e., when the non-linear mode dynamics is dominated by line bending together with non-linear density modulations or convective non-linearities, respectively. The former of these cases has been analyzed in ref. [75] assuming the high\( n \)-expansion, whereas the latter was first studied in ref. [73] in the low-\( n \) limit. Detailed discussions on the other possible non-linear TAE dynamics can be found in the above referenced literature. For the sake of simplicity and of a more compact formalism, we will initially assume \( n \gg 1 \). In this way, we will be able to present a discussion of how the just mentioned mode-mode couplings may alter the local mode structure, eventually leading to saturation.

In the linear limit, the local TAE structure with toroidal mode number \( n \) is given by the coupling of two poloidal harmonics, \( m \) and \( m+1 \), at the radial locations, \( r_0 \), where these modes are degenerate, i.e., \( q(r_0) = (2m+1)/(2n) \). In toroidal coordinates, these modes may be represented as

\[
\delta \phi_{m,n} = e^{-i(\omega_0 t - n \varphi + m \theta)} \left( \frac{T_0 + T_1}{e} \right) \Phi(nq - m, t),
\]

where \( \omega_0 = \omega_A/(2q) = v_A/(2qR_0) \) and \( \Phi(nq - m, t) \) varies on a time scale \( \approx (\epsilon \omega_0)^{-1} \).

**3'2. Non-linear density modulation.** Assume, first, that non-linear effects of ponderomotive force dominate in eqs. (147) and (148). Moreover, assume that the plasma
\( \beta \) is sufficiently small to neglect the ballooning-interchange term in eq. (147) \[57\]. As a consequence of mode couplings, there will be forced low frequency pressure perturbations, which may be obtained from the non-linear (perpendicular) force balance:

\[
\nabla _\perp \left( \delta P_a + \frac{\delta B^2}{8\pi} \right) = 0 .
\]

Meanwhile, assuming an isothermal process, the quasi-neutrality condition along with eq. (149) yields

\[
\frac{\delta \rho_a}{\rho_a} = - \frac{c^2}{4\pi n(T_e + T_i)\omega_0^2} \left| \nabla _\perp (\hat{\mathbf{b}} \cdot \nabla \delta \phi) \right|^2 .
\]

Here, we have also used eq. (148) to the required order, which, for a TAE of frequency \( \omega_0 \) reads \( \delta A_{||} = -i(c/\omega_0)\hat{\mathbf{b}} \cdot \nabla \delta \phi \). Since we are interested in the local non-linear modifications to the TAE mode structure, we concentrate on the region \( nq - m \approx 1/2 \), where, from linear theory, it is well known that only \( m \) and \( m + 1 \) mode numbers contribute, i.e., only \( \delta \overline{\mathbf{F}}(nq - m, \tau) \) and \( \delta \overline{\mathbf{F}}(nq - m - 1, \tau) \) need to be taken into account. Using the fact that \( \delta \overline{\mathbf{F}}(z, \tau) \) is expected to have narrow mode structures about \( z = \pm 1/2 \), and to vary on a time scale \( \approx (\epsilon \omega_0)^{-1} \), eq. (147) along with eq. (148) and eq. (150) can be rewritten as:

\[
(i\partial_t - \nu) \partial_\nu \delta \overline{\mathbf{F}}(\nu + 1/2, \hat{\mathbf{t}}) + (\epsilon_0/4) \partial_\nu \delta \overline{\mathbf{F}}(\nu - 1/2, \hat{\mathbf{t}}) + \frac{b_\nu s^2}{4} (\partial_\nu \delta \overline{\mathbf{F}}(\nu - 1/2, \hat{\mathbf{t}}))^2 \partial_\nu \delta \overline{\mathbf{F}}(\nu + 1/2, \hat{\mathbf{t}}) = C_1 ,
\]

\[
(i\partial_t + \nu) \partial_\nu \delta \overline{\mathbf{F}}(\nu - 1/2, \hat{\mathbf{t}}) + (\epsilon_0/4) \partial_\nu \delta \overline{\mathbf{F}}(\nu + 1/2, \hat{\mathbf{t}}) + \frac{b_\nu s^2}{4} (\partial_\nu \delta \overline{\mathbf{F}}(\nu + 1/2, \hat{\mathbf{t}}))^2 \partial_\nu \delta \overline{\mathbf{F}}(\nu - 1/2, \hat{\mathbf{t}}) = C_2 .
\]

Here, \( \hat{\mathbf{t}} \equiv \omega_A t/q, \nu \equiv nq - m - 1/2, b_\nu \equiv (m/r_0)^2(T_e + T_i)/m\omega_0^2, \hat{s} = r_0 q'/q \) and \( C_1 \) and \( C_2 \) are integration constants, which must be provided by the matching with the linear solution \[44\]. It is possible to show \[36\] that \( C_1 \) and \( C_2 \) can be expressed as a function of the ballooning representation \[57\] of \( \delta \overline{\mathbf{F}}(z, \hat{\mathbf{t}}) \):

\[
\hat{\delta \overline{\mathbf{F}}}(\theta, \hat{\mathbf{t}}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta \overline{\mathbf{F}}(z, \hat{\mathbf{t}}) e^{-i\omega z} dz = \frac{A_{c}(\theta_1) \cos(\theta_0/2) + A_{s}(\theta_1) \sin(\theta_0/2)}{(1 + \hat{s}^2 \theta_1^2)^{1/2}} ,
\]

where \( \theta_0 = 1 \) and \( \theta_1 \approx 1/\epsilon_0 \) (eq. (152) is analogous to eq. (117)). It is readily demonstrated that \[36\]

\[
C_1 = -C_2 = \frac{1}{\sqrt{2\pi}} \frac{A_{c_0}}{\hat{s}} ,
\]

\[
\int_{-\infty}^{\infty} \partial_\nu \delta \overline{\mathbf{F}}(\nu - 1/2, \hat{\mathbf{t}}) d\nu = - \int_{-\infty}^{\infty} \partial_\nu \delta \overline{\mathbf{F}}(\nu + 1/2, \hat{\mathbf{t}}) d\nu = \frac{\pi}{\sqrt{2\pi}} \frac{A_{s_0}}{\hat{s}} ,
\]

where \( A_{c_0} = A_c(\theta_1 = 0) \) and \( A_{s_0} = A_s(\theta_1 = 0) \). The quantities \( A_{c_0} \) and \( A_{s_0} \) are related by the TAE local dispersion relation which imposes

\[
\frac{A_{c_0}}{A_{s_0}} = \delta T_f ,
\]

with \( \delta T_f \) being a function of equilibrium profiles related to the MHD (fluid) potential energy of the system.
Equations (151) can be put in a different form, which helps identifying the typical mode amplitude at saturation: with the definitions
\[
H = (\delta / 2)(b_n / \epsilon_0)^{1/2}(\partial_\nu \delta F(\nu - 1/2, \hat{t}) - \partial_\nu \delta F(\nu + 1/2, \hat{t})) ,
\]
\[
K = -(\delta / 2)(b_n / \epsilon_0)^{1/2}(\partial_\nu \delta F(\nu - 1/2, \hat{t}) + \partial_\nu \delta F(\nu + 1/2, \hat{t})) ,
\]
\[
\zeta = 4\nu / \epsilon_0 ,
\]
\[
\bar{A} = 4(\Lambda_0 / \sqrt{2\pi})(b_n / \epsilon_0)^{3/2} ,
\]
they become [77]
\[
\begin{align*}
(\Lambda + 1) K - \zeta H + (K^2 - H^2)K &= 0 , \\
(\Lambda - 1) H - \zeta K - (K^2 - H^2)H &= -\bar{A} .
\end{align*}
\]
(155)

Here, a time dependence \( \approx \exp - (i\lambda \hat{t}) \) has been assumed for the mode at saturation, and the quantity \( \Lambda = 4\lambda / \epsilon_0 \) has been introduced. Equations (155) is closed by the local dispersion relation, eq. (154), which becomes, together with eq. (153):
\[
\int_{-\infty}^{\infty} H (\zeta, \bar{A}) \, d\zeta = \pi \bar{A} \delta T_I ,
\]
which may be intended as an implicit definition of the non-linear frequency as a function of \( \bar{A} \) and parameterized by \( \delta T_I \), i.e., \( \Lambda = \Lambda_{NL}(\bar{A} \delta T_I) \).

Non-linear saturation occurs due to the appearance of singularities in the mode structure above a critical mode amplitude, \( \bar{A}_c \), as discussed in refs. [44, 72, 73]. In fact, above \( \bar{A}_c \), it can be readily verified that a regular solution of eq. (155), which asymptotically (\( |\zeta| \gg 1 \)) matches the linear solution, no longer exists, because of the appearance of bifurcations in the mode structure, i.e., of fine scales which eventually yield mode saturation in the presence of finite dissipation. It is possible to explicitly derive an analytical expression for \( \bar{A}_c \) parameterized with \( \Lambda \) [77]
\[
\bar{A}_{c1} = \frac{4}{3} \left[ 1 - \Lambda + (\Lambda^2 + \Lambda + 1)^{1/2} \right] \left[ \frac{\Lambda - 1}{6} + \frac{(\Lambda^2 + \Lambda + 1)^{1/2}}{3} \right]^{1/2} ; \ \Lambda < 1
\]
and
\[
\bar{A}_{c2} = \frac{2(\Lambda - 1)^{3/2}}{3\sqrt{3}} ; \ \Lambda \geq 1
\]
(157)

Clearly, the actual value of \( \bar{A}_c(\delta T_I) \) is obtained by the simultaneous solution of the non-linear dispersion relation, eq. (156), together with eq. (157). Figure 40 shows \( \Lambda = \Lambda_{NL}(\bar{A} \delta T_I) \) as a function of \( \bar{A} \). The values of \( \bar{A}_{c1} \) and \( \bar{A}_{c2} \) of eq. (157) are also shown. The intersection of the \( \Lambda = \Lambda_{NL}(\bar{A} \delta T_I) \) curves with those representing \( \bar{A}_{c1} \) and \( \bar{A}_{c2} \) provide the actual functional form of \( \bar{A}_c(\delta T_I) \), which is shown in fig. 41. At saturation, the value of \( A_{c0} \) is expected to be \( A_{c0} = (\sqrt{2\pi} / 4)(\epsilon_0^3 / b_n)^{1/2} \bar{A}_c \), i.e., to exhibit a \( (\epsilon_0^3 / b_n)^{1/2} \) scaling. Numerical solutions of eq. (147) [77], neglecting the ballooning-interchange term and keeping only the \( \delta Q_n / \epsilon_0 \) non-linearity given in eq. (150), provide direct confirmation of the theoretically expected scalings. In fact, assuming \( \hat{s} = 1 \) as a fixed parameter, fig. 42 indicate that both the \( \epsilon_0^{3/2} \) scaling (at fixed \( b_n \)) and the \( b_n^{-1/2} \) scaling (at fixed \( \epsilon_0 \)) are well reproduced. Also the magnitudes of numerical solutions at saturation compare well with the analytical predictions [77]: for \( \epsilon_0 = 0.2, \hat{s} = 1.0 \) and \( b_n = 1.0 \), the numerical saturation is \( A_{c0} = 5.2 \times 10^{-3} \); meanwhile, the analytical estimate yields \( A_{c0} = 5.32 \times 10^{-3} \).
Fig. 40. – The quantity $\Lambda = \Lambda_N \left( \bar{A}, \delta T_f \right)$ as a function of $\bar{A}$ is shown. Different curves correspond to different values of $\delta T_f$, increasing from bottom to top from $\delta T_f = 0.4$ to $\delta T_f = 4.0$ with increments of 0.4. The values of $\bar{A}_{c1}$ and $\bar{A}_{c2}$ of eq. (157) are also shown.

Fig. 41. – Functional dependence of $\bar{A}_c(\delta T_f)$ on $\delta T_f$. 
Fig. 42. – Scalings of $A_{\epsilon_0}$ at mode saturation, in the case $\delta = 1$. The $\epsilon_0^{3/2}$ scaling is obtained with fixed $b_s = 1$; the $b_s^{-1/2}$ scaling with fixed $\epsilon_0 = 0.2$. 
Present results allow us to give estimates of the mode amplitude at saturation. In fact, the typical solution of eq. (155) gives \( H \approx K \approx \hat{A} \). This, together with the definition of \( H \), yields, at mode saturation, \( \partial_r \delta \Phi = \left( \epsilon_0 / b_s \right)^{1/2} s^{-1} \hat{A} \), i.e.,
\[
(\delta B_r / B_0) \approx \hat{A} \left( \epsilon_0 / b_s \right)^{1/2} \beta^{1/2} \partial_r \delta \Phi \approx \left( \epsilon_0 / b_s \right)^{3/2} \beta^{1/2} \hat{A} \left( \delta T_f \right).
\]
This result confirms the fact that the efficiency of mode-mode couplings in determining the TAE amplitude at saturation greatly depends on the location of the linear mode frequency within the toroidal gap, i.e., on the magnitude of the function \( \hat{A}_c(\delta T_f) \), which may be vanishingly small [44, 72, 73, 77].

3.3. \( E \times B \) convective non-linearities. Assume, now, that non-linear line bending together with convective non-linearities dominate in eqs. (147) and (148). As a consequence of mode couplings, there will be forced non-linear oscillations in the scalar and vector potential, which may be shown to be given (in the high-n limit) [44, 72, 73] by:
\[
\delta \dot{\phi}_{1,0} = \frac{c}{\omega_0 B r_0} \frac{m}{r} \delta \hat{\phi}_{n,m} \delta \hat{\phi}_{m+1,n} ,
\]
\[
\frac{\partial}{\partial r} \delta \phi_{2m+1,2n} = -\frac{c}{\omega_0 B r_0} \frac{m}{r} \left( \frac{2}{\omega_0 B r_0} \delta \hat{\phi}_{m,n} \frac{\partial}{\partial r} \delta \phi_{m+1,n} - \delta \hat{\phi}_{m,n} \frac{\partial}{\partial r} \delta \phi_{m+1,n} \right) ,
\]
(158) \[
\frac{\partial}{\partial r} \delta \dot{A}_{1,0} = -\frac{c^2}{\omega_0 B v_A r_0} \frac{m}{r} \left( \frac{2}{\omega_0 B v_A r_0} \delta \hat{\phi}_{n,m} \frac{\partial}{\partial r} \delta \phi_{m+1,n} - \delta \hat{\phi}_{n,m} \frac{\partial}{\partial r} \delta \phi_{m+1,n} \right) ,
\]
(159) \[
\delta \dot{A}_{2m+1,2n} = -\frac{c^2}{\omega_0 B v_A r_0} \frac{m}{r} \left( \frac{2}{\omega_0 B v_A r_0} \delta \hat{\phi}_{m,n} \frac{\partial}{\partial r} \delta \phi_{m+1,n} - \delta \hat{\phi}_{m,n} \frac{\partial}{\partial r} \delta \phi_{m+1,n} \right) .
\]

When substituted back into eqs. (147) and (148), eqs. (158) and (159) yield the following coupled equations for the fields \( \delta \Phi(\nu \pm 1/2, \hat{t}) \)
\[
(i \partial_t - \nu) \partial_r \delta \Phi(\nu + 1/2, \hat{t}) + (\epsilon_0 / 4) \partial_t \delta \Phi(\nu - 1/2, \hat{t}) +
\]
\[
- 2m^2 q^2 s^2 \left( \frac{R_0}{r_0} \right)^2 \beta b_s \delta \Phi(\nu - 1/2, \hat{t}) \right)^2 \partial_r \delta \Phi(\nu + 1/2, \hat{t}) = \frac{1}{\sqrt{2} \pi} \frac{A_{n0}}{s} ,
\]
\[
(i \partial_t + \nu) \partial_r \delta \Phi(\nu - 1/2, \hat{t}) + (\epsilon_0 / 4) \partial_t \delta \Phi(\nu + 1/2, \hat{t}) +
\]
\[
- 2m^2 q^2 s^2 \left( \frac{R_0}{r_0} \right)^2 \beta b_s \delta \Phi(\nu + 1/2, \hat{t}) \right)^2 \partial_r \delta \Phi(\nu - 1/2, \hat{t}) = -\frac{1}{\sqrt{2} \pi} \frac{A_{n0}}{s} ,
\]
which closely resembles eq. (151). Introducing the scaled fields
\[
U = 8 \sqrt{2} \mu q s \left( \frac{R_0}{r_0} \right) \left( \frac{\beta b_s}{\epsilon_0} \right)^{1/2} \delta \Phi(\nu + 1/2, \hat{t}) ,
\]
\[
V = 8 \sqrt{2} \mu q s \left( \frac{R_0}{r_0} \right) \left( \frac{\beta b_s}{\epsilon_0} \right)^{1/2} \delta \Phi(\nu - 1/2, \hat{t}) ,
\]
eqs. (160) become:
\[
(i \partial_r - \zeta) \partial_t U + \partial_r V - \partial_t V^n \partial_r U = \vec{B} ,
\]
(161)
\[
(i \partial_r + \zeta) \partial_t U + \partial_r V - \partial_t V^n \partial_r V = -\vec{B} .
\]
Here, $\tau = \epsilon_0 \delta/4$ and
\[
\vec{B} = 8mq \left( \frac{R_0}{r_0} \right) \left( \frac{\beta b_c}{\epsilon_0} \right)^{1/2} \frac{A_{ci}}{\sqrt{\pi}}.
\]
In the present case, the analysis can proceed in the same fashion as in the case of eq. (151). In particular, from eq. (161), it is possible to give estimates of the mode amplitude at saturation, i.e.,
\[
\delta B_c / B_0 \approx s b_s^{1/2} \beta^{1/2} \partial_\phi \delta \mathbf{\Phi} \approx \epsilon_0^{3/2} \left( \frac{r_0}{R_0} \right) \left( \frac{1}{mq} \right) \vec{B}_c(\delta T_f) \approx \epsilon_0^{5/2} \frac{\vec{B}_c(\delta T_f)}{mq},
\]
where $\vec{B}_c(\delta T_f)$ is the value of $\vec{B}$ at saturation and can be shown to have a dependence on $\delta T_f$ similar to that of the quantity $\vec{A}_c(\delta T_f)$ [44, 72, 73].

Incidentally, we note that the scaling of eq. (162) is valid also in the low-$n$ limit [44, 72, 73] and, in fact, in the remainder of this section we will discuss such limit. This will allow us to directly compare the analytical results, obtained with the method presented above and specialized to the low-$n$ case, with those of a direct numerical solution of the reduced MHD equations, eqs. (170) and (183) of the appendix sects. 5'3 and 5'4, respectively. More precisely, we will analyze the non-linear dynamics of $n = 1$ global modes induced by the toroidicity, namely the TAE and the RPSAE.

As already stated in the Introduction, these modes can be driven unstable by the resonant interaction of the mode with energetic ions, such as $\alpha$-particles produced in fusion reactions. To describe this effect properly, a kinetic description of the energetic particle population would be necessary. With the aim of studying only the saturation mechanisms related to the MHD non-linearities, however, it will be sufficient to include a model forcing term in the reduced MHD equations in order to drive the system from the linear to the non-linear regime.

To separately study the dynamics of the different global modes, we need to selectively drive only one of them at a time. To this purpose, we can use the fact that both TAE’s and RPSAE’s eigenfunctions are localized around the gap region, whereas GAE’s have eigenmode structures which tend to peak towards the edge of the plasma column. Thus, localizing the forcing term in different radial regions will help in driving GAE’s or gap modes. In order to further discriminate between TAE and RPSAE, we can look at the relative phase of the Fourier components of their eigenfunctions [30]. Indeed, as shown in figs. (21) and (25), TAE’s are characterized by in phase components, whereas the (upper) RPSAE’s by out of phase ones.

The model forcing term used in eq. (183) of appendix 5'4 has Fourier harmonics:
\[
(F_{ac})_{m,n} = \hat{\partial} \gamma_{ac} \left( \frac{\nabla^2 \phi}{\phi} \right)_{m,n},
\]
where $F_{ac}$ replaces the term proportional to $\nabla \cdot \mathbf{H}_H$, and $\gamma_{ac} = \gamma_{ac}(r)$. In order to drive a gap mode, without destabilizing GAE’s, we choose a bell-shaped radial profile for the function $\gamma_{ac}$, centered at the gap position $r_g$. In particular, the TAE will be driven by the following choice of $(F_{ac})$:
\[
(F_{ac})_{1,1} = \hat{\partial} \gamma_{ac} \left( \frac{\nabla^2 \phi}{\phi} \right)_{1,1}, \quad (F_{ac})_{2,1} = \hat{\partial} \gamma_{ac} \left( \frac{\nabla^2 \phi}{\phi} \right)_{2,1},
\]
whereas the (upper) RPSAE is driven assigning:
\[
(F_{ac})_{1,1} = -\hat{\partial} \gamma_{ac} \left( \frac{\nabla^2 \phi}{\phi} \right)_{2,1}, \quad (F_{ac})_{2,1} = \hat{\partial} \gamma_{ac} \left( \frac{\nabla^2 \phi}{\phi} \right)_{2,1}.
\]
From a theoretical analysis [44] it is possible to show that two time scales exist in the system. The shorter scale corresponds to the oscillation at frequency $\omega = \omega_A/(2g(r_0))$, while the longer one is determined by the $O(\epsilon)$ toroidal coupling and by non-linear effects, and scales as $(\epsilon \omega_A)^{-1}$. Therefore, $\gamma_{ac}/(\epsilon \omega_A)$ is the relevant normalization of the linear drive.

At the gap location, the “fundamental” linear components, $\hat{F}_{m,n}$ and $\hat{F}_{m+1,n}$ (where $F$ generically stays for $\phi$ or $\psi$), generate non-linear $\hat{F}_{1,0}$ beat components (forced by $\hat{F}_{m+1,n} \times \hat{F}_{m,n}$) and $\hat{F}_{2m+1,2n}$ ones (forced by $\hat{F}_{m+1,n} \times \hat{F}_{m,n}$).

An example of a non-linear simulation of TAE is shown in fig. 43. We include in

![Graph showing $W_{TOT,m,n}$ vs. $\omega_A t$ for different Fourier components $(m,n)$](image)

Fig. 43. - Volume integrated energy (magnetic plus kinetic) for different Fourier components $(m,n)$ vs. time for a non-linear simulation of an unstable driven TAE. The $q$-profile has a parabolic radial dependence with $q(0) = 1.1$ and $q(a) = 1.9$. The inverse aspect ratio is $\epsilon = 0.075$, the density is constant $\hat{\rho} = \hat{\rho}_0$ and the resistivity corresponds to $S^{-1} = 10^{-5}$.

the simulation the Fourier components with mode numbers $(1,0), (1,1), (2,1)$ and $(3,2)$. The $q$-profile has a parabolic radial dependence with $q(0) = 1.1$ and $q(a) = 1.9$. The inverse aspect ratio is $\epsilon = 0.075$, the density is uniform $\hat{\rho} = \hat{\rho}_0$, the resistivity corresponds
to $S^{-1} = 10^{-5}$ and a small amount of perpendicular viscosity is added in the vorticity equation (eq. (183)) to improve the numerical stability in the non-linear regime, thus allowing for longer time step. After a transient phase during which the TAE emerges from the initial conditions, the forcing term drives the system in the linear growing phase, with a normalized linear growth rate $\gamma_{\alpha c}/(\epsilon \omega_A) \approx 1.7 \times 10^{-1}$ (the smallness of such parameter ensures that the linear drive can be treated as a perturbative one). The frequency spectrum comes out to be essentially monochromatic, thus showing that only a single eigenmode is driven, with frequency $\omega = \omega_0 \approx 0.33 \omega_A$ on the (1,1) and (2,1) components. When the non-linear modifications to the (1,0) harmonic become significant ($\omega_A t \geq 400$), the system enters the non-linear regime, the growth slows down and saturation is eventually obtained. It has to be noted that, at the lowest order, the (1,0) Fourier component shows a zero frequency perturbation, as expected from theory [44], whereas the (3,2) perturbation exhibits a frequency $\omega \approx 2 \omega_0$.

In fig. 44 a blow-up of the continuous spectrum around the gap position is shown. The spectra obtained in the linear limit and at the beginning of the non-linear phase are compared. As the system enters the non-linear phase, the structure of the gap shrinks, and the mode is forced to have a real frequency inside the lower continuum. The coupling with the KAW’s becomes stronger and the enhanced resistive damping yields saturation [44].

The saturated amplitudes of the fluctuating magnetic field are predicted to scale as $\delta B_{r(1,1)}/B_0 \propto \delta B_{r(2,1)}/B_0 \propto \epsilon^{5/2}$ and $\delta B_{r(1,0)}/B_0 \propto \delta B_{r(3,2)}/B_0 \propto \epsilon^3$. In fig. 45 the numerical results for such amplitudes are reported versus $\epsilon$. The forcing term has been scaled keeping the dimensionless parameter that measures the driving term, $\gamma_{\alpha c}/(\epsilon \omega A)$, constant. Theoretical results [44] are also shown for comparison. A best-fit analysis of the

![Graph](image_url)

Fig. 44. – Blow-up of the Alfvén continuum for the simulation of fig. 43. The continuous spectra obtained in the linear limit and at the beginning of the non-linear phase are compared.
simulation findings gives $\delta B_{r(1,1)}/B_0 \propto \varepsilon^{2.44}$, $\delta B_{r(2,1)}/B_0 \propto \varepsilon^{2.14}$, $\delta B_{r(1,0)}/B_0 \propto \varepsilon^{2.24}$ and $\delta B_{r(3,2)}/B_0 \propto \varepsilon^{2.92}$.

Non-linear MHD saturation has also been found for the RPSAE. To compare the numerical results for the saturated amplitude with the corresponding TAE scaling, also $S^{-1}/\varepsilon^3$ has been kept constant, thus fixing the relative importance of resistive dissipation. The simulation results for the saturation amplitudes, reported in fig. 46, show that the upper RPSAE saturates at a higher level than the TAE, and with a weaker $\varepsilon$ dependence ($\delta B_{r(1,1)}/B_0 \propto \varepsilon^{1.9}$, $\delta B_{r(2,1)}/B_0 \propto \varepsilon^{2.1}$).

3.4. Kinetic non-linearities. – A complementary approach to the TAE saturation problem consists in investigating non-linear wave-particle interactions. In the present section, in fact, we focus on such problem, mainly presenting the results obtained with the Hybrid MHD-Gyrokinetic Code, although the first investigations on this topic have been carried out by other authors both analytically and with different numerical codes. Numerical simulations based on a Hamiltonian guiding-center representation of the particle motion have been indeed performed by Wu and coworkers [67, 68] and Borba et al. [69], while Todo et al. [78] have developed a hybrid MHD-Vlasov code, which, unlike the particle codes (Lagrangian in nature), computes the energetic-particle distribution function in an Eulerian frame. The results found by these authors appear to be in reasonable agreement with those based on a theoretical model [43], which assumes that saturation is due to non-linear trapping of the resonant energetic particles in the potential well of the wave. This model can be summarized as follows. Let $\Psi_{m,n}$ be the wave phase for a particle at the position $(r, \theta, \varphi)$ at the time $t$:

$$\Psi_{m,n} \equiv \omega t - m\theta + n\varphi.$$
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![Graph showing saturated amplitudes versus ε for different Fourier components of the upper RPSAE. The q profile and density are the same as those of fig. 43, whereas the resistivity is scaled according to $S^{-1}/\epsilon^3 = 2.37 \times 10^{-4}$.]

In the linear case (i.e., with unperturbed particle orbits), the resonant energy exchange between the wave and the particle can take place only in a radial shell centered around the resonant surface and qualitatively defined by the condition

\[ \frac{\partial \Psi_{m,n}}{\partial t} \lesssim \gamma_L, \]

where $\gamma_L$ is the linear growth rate of the wave.

When non-linear wave-particle interactions are included, two different kind of orbits exist: closed trajectories for particles trapped in the potential well of the wave and open ones for untrapped particles. Trapped particles are subtracted to the drive, because the back-and-forth motion along the $\Psi_{m,n}$ axis causes their energy exchange with the wave to be averaged to zero. The separatrix between the two families of orbits corresponds to the maximum radial extension of the particle motion, which can be shown to be proportional to the square root of the wave amplitude. Thus, as a linear instability grows, the growth rate of the wave becomes weaker, since more and more particles become trapped. When the separatrix width becomes of the same order of the width of the resonant shell defined by eq. (165), the wave eventually saturates. This condition corresponds to a relationship between the saturation amplitude of the fluctuating radial magnetic field and the linear growth rate. Berk and Brézmann found [43], as it straightforwardly follows from the previous discussion,

\[ \frac{\delta B_r}{B_0} \propto \gamma_L^2. \]
Here, it is worth to recall that particle effects are treated perturbatively: they affect the growth rate of the wave, but have no influence on the real frequency and the mode structure, which are thought to be fixed and given by the dynamics of the core plasma. This approach is fully consistent with that of refs. [67, 68, 69], which, in fact, obtain results in agreement with eq. (166). From the discussion of the previous section, we expect that this perturbative approach is indeed adequate for investigating the saturation of the low-γ_L gap modes (TAE's) or their kinetic counterpart (KTAE's). The results obtained in ref. [78] in the framework of fully self-consistent (non-perturbative) simulations of the gap-mode regime do not significantly depart from the findings of the model proposed in ref. [43] and confirm this expectation. On the contrary, we have to presume that the (high-γ_L) EPM's, for which particle dynamics plays an intrinsically non-perturbative role, can be characterized by a completely different phenomenology, thus requiring a more complete approach. This issue is further discussed in the following, by comparing the findings of self-consistent (i.e., full MHD-gyrokinetic) simulations with those obtained from perturbative non-linear studies, which retain the effects of non-linear wave-particle interactions but neglect the possible modifications of the mode structure and frequency.

3.4.1. Gap-mode saturation

The maximum amplitude of the radial component of the fluctuating magnetic field, obtained at different values of β_H(0) is plotted, in fig. 47, versus the corresponding values of the linear-phase growth rate γ_L. Full boxes refer to perturbative simulations, empty boxes to self-consistent ones. The fair agreement between the two sets of results, all
relative to low-$\gamma_L$ gap modes, confirms the validity of the perturbative approach for this regime. Saturation can be attributed to the trapping of resonant particles, discussed above. This can be easily recognized by examining the findings of a typical perturbative simulation in such regime. Figure 48 shows the time behaviour of the mode amplitude for $\beta_H(0) = 0.08$, with a given background dissipation represented by a damping rate $\gamma_D = 0.01\omega_A$, and the other parameters as in the simulations considered in fig. 36. Such

\[ \ln A(t) \]

\[ \begin{array}{c}
\text{In } A(t) \\
\hline
\text{0} & \text{200} & \text{400} & \text{600} & \text{800} \\
\text{0} & 2 & 4 & 6 \\
\end{array} \]

$\omega_A t$

Fig. 48. – Time evolution of the mode amplitude $A(t)$ for a perturbative non-linear simulation with $\gamma_D = 0.01\omega_A$, $\beta_H(0) = 0.08$, and the other parameters as in fig. 36.

case corresponds, in fig. 47, to the point relative to $\gamma_L = 0.03\omega_A$. After the initial linear-growth phase, the non-linear growth rate progressively decreases and saturation is eventually reached. The time evolutions of the radial coordinate $r$ and the parallel velocity $U$ are reported in fig. 49(a) and (b), respectively, for a particle initially close to the resonance condition – eq.(165) – with the $m = 2$ poloidal component. It can be noted that during the linear phase ($\omega_A t < 150$) the particle orbit is almost unperturbed, the oscillation of the particle radial position being due to the effect of magnetic drifts, while the significant decrease of the non-linear growth rate coincides with a strong modification of the original linear orbit. The trajectories in the $(\Psi_{21}, r)$ plane, for the same particle, are shown in fig. 50, for the time intervals $0 < \omega_A t < 284$ (a) and $264 < \omega_A t < 560$ (b) (note that the two intervals are slightly overlapping). For the sake of clarity, the $\Psi_{21}$ axis is mapped onto the interval $0 < \Psi_{21} < 4\pi$. It is evident that, in the first interval (linear growth of the wave), the trajectory corresponds to that of a passing particle, whilst, when the mode amplitude reaches a certain level, the particle becomes trapped, stops driving the mode and, in fact, contributes to its saturation.

The results of the self-consistent simulations allow us to draw, as expected, similar conclusions. In fig. 51, the time evolution of the poloidal harmonics of the total fluctuating energy for the KTAE obtained at $\beta_H(0) = 0.02$ (corresponding to $\gamma_L/\omega_A = 0.023$)
Fig. 49. — Time evolution of the radial coordinate (a) and the parallel velocity (b) of a nearly-resonant (with respect to the component \( m = 2 \)) energetic particle, for the case of fig. 48. During the linear-growth phase the orbit remains almost unperturbed. The saturation phase coincides with a significant alteration of the particle motion.

Fig. 50. — Orbit in the plane \( (\Psi_{2,1}, r) \), for the same particle as in fig. 49, in the time intervals \( 0 < \omega_A t < 284 \) (a) (linear growth) and \( 264 < \omega_A t < 500 \) (b) (non-linear saturation). The \( \Psi_{2,1} \) axis is mapped onto the interval \( 0 \leq \Psi_{2,1} < 4\pi \). The particle is initially passing, but becomes trapped as the mode reaches a certain amplitude.
is plotted. Figure 52 shows, for the same case, the normalized energetic-particle line-

density profile ($\propto \rho n_H(r)$) at two different times: the solid line refers to $\omega_A t = 30$, 
during the linear growth of the mode; the dashed line to $\omega_A t = 480$, after saturation 
has been reached. No appreciable modification in the density profile can be observed, 
consistently with the conjecture of “soft” trapping mechanism.

Two different sub-regimes can be distinguished with respect to the scaling of the 
saturated amplitude. In the first sub-regime, at very low values of $\gamma_L$, a dependence 
$\delta B_r|_{sat} \propto \gamma_L^\alpha$, with $\alpha \leq 4$, is obtained. Note that such a dependence differs from that [43] 
reported in eq. (166) and obtained also in refs. [67, 68, 69]. It is possible to guess that the 
finite character of the growth rates considered here does not allow us a straightforward 
comparison with the vanishing-$\gamma_L$ approach of ref. [43]. The second sub-regime can 
be identified at slightly higher (although still low) values of $\gamma_L$. The $\gamma_L$ dependence of 
$\delta B_r|_{sat}$ tends to become weaker as the radial width of the resonant region (proportional 
to $\gamma_L$) exceeds the (finite) radial width of the mode, and the fraction of resonating 
particles is consequently cut off. Saturation is then reached at almost constant values of 
$\delta B_r|_{sat}$, irrespectively to the value of $\gamma_L$. 

Fig. 51. – Time evolution of the poloidal harmonics of the total fluctuating energy for a $n = 1$, 
$\beta_H(0) = 0.02$ self-consistent non-linear simulation.
Fig. 52. – Energetic-particle line-density profile, for the case shown in fig. 51, at $\omega_\lambda t = 30$ (solid line), during the linear growth of the mode, and at $\omega_\lambda t = 480$ (dashed line), after that saturation has been reached. No appreciable modification can be observed.

3.4.2. EPM saturation

On the basis of the results presented in sect. 2.6.3 we know that EPM saturation cannot be investigated within a perturbative approach and that it requires self-consistent simulations. Figure 53(a) shows the time evolution of the total energy for such a simulation, with $\beta_H(0) = 0.04$, while the contour plot of the line-density in the $(t,r)$ plane is represented in fig. 53(b). We note that, unlike the case examined in the previous section, saturation occurs because of a sudden displacement of the energetic-particle population (the instability source). This displacement is also evident in fig. 54, which shows the line-density profile before (solid line) and after (dashed line) saturation, and can be attributed to the secular drift that affects the radial motion of energetic particles. Such a drift can be observed, for a particle with initial radial coordinate $r = 0.4 a$, in fig. 55.

The secular radial drift induced on the circulating particles in resonance with the wave is analogous to that discussed, for the fishbone mode, in ref. [79]. In particular, it is evident, from fig. 53, that the secular motion becomes larger at larger mode amplitude, in agreement with the process called “mode-particle pumping” in ref. [80]. We must however remind that fishbone oscillations have $n = 1$ and they are typically driven unstable by resonant interactions with magnetically-trapped energetic particles, whereas the most unstable EPM’s are characterized by larger $n$ values, and those described here are destabilized by circulating energetic ions.

The fact that the secular radial motion becomes negligible for particles located at radial positions where the mode amplitude is small causes the formation of a sharp energetic-particle density gradient at the plasma boundary. A consistent description of appreciable particle losses through the plasma boundary would then require to remove the fixed-boundary constraint and include “external” poloidal harmonics (with $m/n > q_a$),
Fig. 53. – Time evolution of the poloidal harmonics of the total energy (a), and contour plot of the line-density in the \((t, r)\) plane (b) for a self-consistent simulation with \(\beta_H(0) = 0.04\) (EPM regime). Saturation occurs because of a macroscopic displacement of the energetic-particle population.
Fig. 54. – Line-density profile at \( \omega_A t = 20 \) (solid line), during the linear growth of the mode, and at \( \omega_A t = 40 \) (dashed line), after saturation, for the simulation shown in fig. 53. A macroscopic displacement of the source of instability is observed.

Fig. 55. – Poloidal-plane projection of a typical energetic-particle orbit in the simulation of fig. 53. The initial radial coordinate is \( r = 0.4a \).
coupled to the fundamental $m = 1, 2$ EPM components. We also note that the inclusion of a significant background dissipation in the model and of a source term in the Vlasov equation would yield “bursting” cycles of the EPM instability and of the associated particle displacement [79, 81] instead of the steady-state saturation described here.

Particle redistributions associated to EPM’s can be appreciated, from a quantitative point of view, in fig. 56, which shows the fraction of the global energetic-particle population displaced out of a fixed radial position ($r = 0.7a$, in this case). Results relative to simulations with toroidal number $n = 1$ (full boxes) and $n = 4$ (empty boxes) are plotted for different values of $\beta_H(0)$. These findings forbid to generalize to the whole family of Alfvén modes the results concerning gap modes, concluding, on the basis of perturbative treatments, that such family has little influence on the transport and confinement of energetic particles.

The inclusion of MHD non-linearities is not expected to appreciably modify these results. Indeed, it has been shown [44, 72] that such non-linearities mainly alter the shear Alfvén continuum in the region close to the gap. Such modification can influence the continuum damping of gap modes and, eventually, yield their saturation. The existence of EPM’s and their dynamics are weakly affected by the presence of the gap; thus, we can guess that the EPM saturation mechanism, described here, remains effective even if the MHD non-linear terms are fully taken into account in the simulation. This is confirmed by the results shown in figs. 57 to 60, where two simulations are compared, with and without MHD non-linearities, respectively. No significant difference can be appreciated for both the saturation level (see figs. 57a and 59a) and the particle-displacement mechanism (see figs. 57b and 59b, as well as figs. 58 and 60).
Fig. 57. – Time evolution of the $n = 1$ poloidal components of the total energy (a), and contour plot of the line-density in the $(t, r)$ plane (b) for a simulation analogous to that shown in fig. 53, but including the MHD non-linearities. An artificial viscous term has been introduced in the fluid equations and a larger resistivity value has been fixed ($S^{-1} = 10^{-3}$) in order to prevent numerical instabilities due to the explicit character of the non-linear MHD terms.
Finally, we note that the $\beta_H$ threshold for the deterioration of confinement properties is characterized by a strong correlation with the EPM destabilization (cf. figs. 34 and 38). Thus, a relevant dependence on the toroidal mode number is also expected, since high-$n$ EPM's are the most unstable ones. Indeed, while in the $n = 1$ case such a threshold appears to be unrealistically high (with the parameters considered in this paper, $\beta_H(0)_{th} \approx 0.024$), for $n = 4$ and $n = 8$ we find $\beta_H(0)_{th} \approx 0.011$ and $\beta_H(0)_{th} \approx 0.006$, respectively (cf. fig. 39). EPM's with $n \gtrsim 10$ have been predicted to be unstable in plasmas close to ignition conditions [58, 82, 83]. High resolution self-consistent simulations are required to check whether the qualitative features of EPM dynamics, discussed here, are preserved in such high-$n$ regime and whether the $\beta_H$ threshold for confinement deterioration of charged fusion products may be significantly low for realistic ignited-plasma conditions.

4. Discussions and Open Problems

The present review article has dealt with both linear stability properties of shear Alfvén waves in tokamaks and their non-linear dynamic evolution. Essentially, the interest in this problem is given by the consequences that these modes may have on energetic particle confinement and, thus, by their potential detrimental effect on the ignition margins of future experimental devices as ITER.

From the overall discussion on the linear stability of the Alfvén spectrum, given in sect. 2, it can be stated, with reasonable confidence, that the present understanding of this issue is thorough and complete, at least qualitatively. In fact, the various numerical codes, available at present, make it possible to precisely compute the linear spectrum of Alfvén eigenmodes in realistic tokamak geometries, e.g., that of ITER, up to toroidal mode numbers $n \gtrsim 10$. Nonetheless, analytical investigations of the most unstable mode
Fig. 59. – Same as fig. 57, without MHD non-linearities. No appreciable difference is observed with respect to the saturation mechanism in the two cases. Note that the simulation differs from that of fig. 53 only for the inclusion of a viscous term and the different resistivity value.
numbers – in the range $10 \lesssim n \lesssim 50$ for ITER – are still required due to memory limitation of present computer facilities. Asymptotic analyses, of the type discussed in section 2.5, offer extremely useful tools in this respect, although their advantage becomes quite questionable if they are to be applied to investigations of realistic plasma equilibria, for which these analytic-theoretical tools become difficult to handle and even to implement numerically. Thus, despite the present fundamental understanding of Alfvén eigenmode stability in tokamaks, there still exist limitations in the capability of predicting the stability properties of these modes in a real device [84].

Besides the problems related to the linear stability of Alfvén eigenmodes in tokamaks, a separate consideration is necessary for EPM’s, which are not eigenmodes of the thermal plasma and whose stability properties, thus, deserve a specific attention. This is true not only because the energetic particle population must be treated non-perturbatively, i.e., on the same footing of the thermal plasma (cf. sect. 2.5.5), but also because EPM’s, as it was recently emphasized, for sufficiently high pressure gradients of the thermal plasma or energetic particle density [42, 63, 85], smoothly connect to the Kinetic Ballooning Mode (KBM) branch [86, 87] resonantly excited by energetic particles. These modes have been observed experimentally and have been given the name of Beta induced Alfvén Eigenmodes (BAE) [88, 89], since their frequency is near (not necessarily inside) the beta induced frequency gap in the shear Alfvén continuum ($0 < (\omega/\omega_A)^2 \lesssim \Gamma\beta$, with $\Gamma$ being the ratio of specific heats; cf. sect. 2). It has been also demonstrated that, when the Alfvén mode frequency becomes so low to be comparable with the ion diamagnetic frequency and to the thermal ions transit frequency ($\approx v_{ti}/(qR_0)$), eigenmodes of the shear Alfvén branch may be excited even in the absence of an energetic particle drive [90]. All these low-frequency Alfvén modes may have serious detrimental effects on the energetic particle confinement [88], analogously to what happens for the higher frequency modes

Fig. 60. – Initial (solid line) and final (dashed line) line-density profile for the simulation shown in fig. 59.
discussed in the present review article. Nonetheless, the study of the former instabilities has not quite received the same attention as the latter, and, thus, they have not been analyzed here. However, it is clear that further studies on this issue are needed, due to its possible impacts on the performances of reactor relevant plasmas.

The present understanding of non-linear dynamic properties of Alfvén modes in tokamaks is, unlike for the fundamental insight into their linear stability, still not entirely satisfactory. The most widely accepted picture of non-linear saturation of Alfvén eigenmodes is a single resonance local theoretical model [43], which assumes that saturation is due to non-linear trapping of the resonant energetic particles in the potential well of the wave. The saturation level, in this case, can be shown to scale as

\[ \delta B_L/B_0 \propto (a/\rho_{LH})(\gamma L/\omega)^2 \propto n(\gamma L/\omega)^2 \];

thus, it is expected to be important near threshold and for relatively small mode numbers. For larger growth rates and/or mode numbers, the saturation mechanism due to wave-particle trapping is eventually substituted by that due to mode-mode couplings, which, e.g., predicts \( \delta B_L/B_0 \approx (1/n)^{5/2} \) in the case of \( \mathbf{E} \times \mathbf{B} \) convective non-linearities [44]. In all these cases, however, the expected particle losses are negligible, unless a threshold in the mode amplitude for stochasticity of particle orbits in phase space is reached, which can be estimated by the Chirikov criterion of resonance overlapping [91]. Meanwhile, both wave-particle trapping and mode-mode coupling based models, discussed in the present review article, are single resonance models and, thus, intrinsically not applicable to a multi resonance scenario with stochastic particle orbits. In this respect, the first work that analyzes gap mode saturation accounting for global mode structures in the high-n case is ref. [71], which yields \( \delta B_L/B \approx c_0^2(\gamma L/\omega)^{1/2} \) for saturation via frequency cascading, due to non-linear ion Landau damping of low-frequency waves. In any event, a general understanding of non-linear gap mode dynamics is, at present, still lacking. This fact, together with the above mentioned uncertainties in predicting the linear Alfvén spectrum in a realistic plasma equilibrium, makes it impossible, to date, to forecast the level of energetic particle losses in a next generation device like ITER. However, it can be argued that the typical ITER condition \(- (\rho_{LH}/a) < 1 - \) should play a beneficial role in keeping energetic particle losses much smaller than in present experimental machines.

As in the case of linear stability, the non-linear dynamics of EPM need a separate discussion. As shown in the present review article and in previous published papers [85, 92], the saturation mechanism of these modes is essentially different from that of gap modes and it is crucially related to macroscopic redistributions of the energetic particle source. As expected, this novel behavior is related to the non-perturbative contribution of particle dynamics to both linear and non-linear evolution of EPM's. This remark also explains why, as shown in sect. 3.4, the threshold in fast particle energy density \( \langle \beta_{TH} \rangle \) for EPM destabilization coincides with the threshold for macroscopic particle redistribution. From the present understanding, we expect that \( \beta_{TH} \) decreases with \( n \) as it is indeed observed in numerical simulations [85, 92] – eventually reaching a minimum for \( n = n_{max} \). Due to memory limitations, numerical simulations have not yet reached high enough mode numbers to show the minimum threshold at \( n_{max} \). However, from high-\( n \) one-dimensional linear gyrokinetic simulations [63] (cf. sect. 3.4), one can predict \( n_{max} \approx R_0/(q_{TH}) \) \( (R_0 \) being the radial location of the mode) with \( \beta_{TH}(0) \approx (0.05/q^2)\left(\rho_{TH}/R_0\right) \) and \( \rho_{TH} \), representing the typical scale length of \( \beta_{TH} \) at threshold. Future investigations need to confirm this theoretically expected values via global non-linear EPM numerical simulations.

In the analyses of non-linear dynamics of Alfvén modes in tokamaks, the aspects
involving low-frequency waves – in the range of thermal ion diamagnetic and transit frequencies – is still untouched. Nonetheless, this topic remains one of the most interesting from a theoretical point of view and one of potentially great practical impact. In fact, since excitations of these low-frequency modes are possible even in the absence of an energetic particle drive [90], their non-linear dynamics may affect the energetic particle confinement properties as well as those of the thermal plasma.

In summary, since the first works on excitations of Alfvén modes in tokamaks by energetic particle populations [7, 8, 9], the theoretical understanding of this problem has made impressive advances to such an extent that, at present, the challenge is to be capable of precisely predicting the properties of Alfvén wave spectra and their possible consequences on particle and energy confinement in real devices. There are many open problems along this route, and we may look forward to new exciting and very likely controversial results.
5. – Appendices

In the following, we will summarize the main features of the numerical instruments used in our analysis. In particular, the equilibrium code CHEASE [93] and the linear stability one, MARS [94, 95], will be briefly described. Then, the derivation of the $O(e^3)$-reduced MHD model is presented, along with the Hybrid MHD-Gyrokinetic Code [72], which is based on such model and allows us to investigate the effects of a population of energetic ions on the stability and non-linear dynamics of shear Alfvén modes.

5.1. CHEASE. – Numerical equilibria in toroidal geometry may be calculated using the CHEASE code [93]. The MHD equilibrium equations read

\[ \nabla P_0 = \frac{1}{c} J_0 \times B_0 , \]

\[ \nabla \times B_0 = \frac{4\pi}{c} J_0 , \]

\[ \nabla \cdot B_0 = 0 . \]

(167)

Assuming axisymmetry, the magnetic field can be represented as

\[ B_0 = F(\psi)\nabla \varphi + R_0 \nabla \psi \times \nabla \varphi , \]

where $\varphi$ is the ignorable toroidal angle, $\psi$ is the poloidal magnetic flux function and $R_0$ is the major radius of the torus, measured at the geometric axis of the plasma vacuum chamber. For the axisymmetric equilibria we are considering, the pressure $P_0$ and the function $F$ depend on $\psi$ only ($P_0 = P_0(\psi)$, $F = F(\psi) \equiv R B_{0\varphi}$). Substituting eq. (168) into eqs. (167) leads to the elliptic second-order non-linear partial differential Grad-Shafranov equation [45]

\[ \Delta^* \psi = 4\pi R_0^2 \frac{dP_0}{R^2 d\psi} + \frac{1}{R_0^2} \frac{dF}{d\psi} , \]

(169)

which completely describes the two-dimensional (2-D) toroidal equilibria once the free functions $P_0(\psi)$ and $F(\psi)$ are given. The elliptic operator $\Delta^*$ is given by

\[ \Delta^* \psi = R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right) = R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2} . \]

(170)

The CHEASE code represents $\psi$ by Hermite cubic elements, allowing very accurate equilibrium calculations with modest mesh sizes. The poloidal angle $\chi$ is determined once fixed the functional dependence of the Jacobian $J = (\nabla \psi \times \nabla \varphi \cdot \nabla \chi)^{-1}$, which in the CHEASE code has the form

\[ J = C(\psi)R_0^\alpha |\nabla \psi|^\mu , \]

where $\alpha$ and $\mu$ are integers. $C(\psi)$ is determined by demanding that $\chi$ increases by $2\pi$ per poloidal turn. The choice of $\alpha = 2$ and $\mu = 0$ has been done in the present paper.

5.2. MARS. – The MHD linear stability in general toroidal geometry can be investigated using the full, toroidal, resistive-MHD code MARS [94, 95], which uses the equilibrium calculated by the CHEASE code as input. MARS solves the linear limit of the MHD eqs. (1-7) with eq. (4) replaced by the resistive version, eq. (10). The problem is solved in flux coordinates $(s, \chi, \varphi)$, where $s$ is a radial coordinate (as, e.g.,
\[ s = (1 - \psi/\psi_{axb})^{1/2}, \text{ with } s = 0 \text{ on the axis of the plasma column and } s = 1 \text{ at the plasma boundary or, similarly, } s = (V(\psi)/V_{tot})^{1/2}, \text{ where } V(\psi) \text{ is the volume enclosed inside the magnetic surface labeled by the value } \psi \text{ and } V_{tot} \text{ is the total plasma volume}, \chi \text{ is a poloidal angle-like variable (see appendix sect. 5)} \text{ and } \varphi \text{ is the toroidal angle. As already stated in sect. 2, in general toroidal geometry the poloidal mode number is no more a good "quantum number", because of the 2-D equilibrium dependence, that is, different poloidal harmonics are coupled together. The perturbed variables in MARS are expanded using Fourier series in the poloidal } (\chi) \text{ and toroidal } (\varphi) \text{ angles and a generalization of finite elements in } s [96]. \text{ The dependent variables are the Fourier harmonics of plasma pressure and contravariant } (s, \chi, \varphi) \text{-components of velocity, perturbed magnetic field and current densities, the latter two multiplied by the jacobian } J. \text{ After discretization, the linearized resistive MHD system takes the form of a complex eigenvalue problem:}

\[(171) \quad AX = \lambda B X,\]

where \( A \) and \( B \) are the matrices of the coefficients of MHD-force and inertial operator respectively, \( \lambda \equiv \gamma - i\omega, \) is the complex eigenvalue (a dependence \( \propto \exp(\lambda t) \) is assumed for the perturbed quantities; \( \gamma \) is the growth rate and \( \omega \) is the real frequency), and \( X \) is the vector of the dependent variables. The eigenvalues are found by a generalization of the inverse vector iteration to complex eigenvalues and eigenvectors [97]. In order to converge to the closest eigenvalue in the complex plane to a certain guess \( \lambda_0, \) eq. (171) is first rewritten as

\[(172) \quad (A - \lambda_0 B)X = (\lambda - \lambda_0)B X.\]

This allows to apply the following standard iteration scheme

\[(173) \quad (A - \lambda_0 B)X_{(n)} = BV_{(n-1)}, \]

\[(174) \quad V_{(n)} = \frac{X_{(n)}}{||X_{(n)}||}.\]

The \( n \)-th approximation to the eigenvalue is then given by

\[(175) \quad \lambda_{(n)} = \lambda_0 + \frac{1}{||X_{(n-1)}||} \frac{\langle X_{(n-1)}^*, X_{(n-1)} \rangle}{\langle X_{(n)}^*, X_{(n)} \rangle},\]

where \( \langle Y^*, Z \rangle \) is the inner product of the complex conjugate of vector \( Y \) with vector \( Z \) and \( ||Z|| \equiv \langle Z^*, Z \rangle^{1/2}. \)

Part of the same machinery of the eigensolver can be easily adapted to solve the initial value (rather than the eigenvalue) linear problem, which formally can be recovered from eq. (171) by replacing \( \lambda \) with a time derivative \( \partial_t \) acting only on the vector \( X. \) The time-discretized evolution equation takes indeed the form

\[(176) \quad (A - \frac{1}{\Delta t} B)X_t = -\frac{1}{\Delta t} B X_{t-\Delta t},\]

which has the same structure of eq. (173) under the conditions of identifying \( \lambda_0 \) with \( 1/\Delta t, \) redefining the matrix on the r.h.s. and skipping the renormalization step, eq. (174). In this paper only fixed-boundary modes, that is, modes characterized by normal components of the perturbed magnetic and velocity fields equal to zero at the plasma boundary, are considered, although the code allows the user to handle also a vacuum region surrounding the plasma.
5.3. Order-$\epsilon^3$ reduced MHD. - We introduce, here, a simplified version of the resistive MHD equations, which will be useful later for the non-linear study of the Alfvén modes, greatly reducing the complexity of the problem. We start from the resistive MHD equations (1-7), with the Ohm’s law generalized by eq. (10).

Since tokamak plasmas are characterized by values of the safety factor $q(r) \approx (r B_\varphi)/(R B_\theta) \approx O(1)$ ($B_\varphi$ and $B_\theta$ are, respectively, the toroidal and poloidal component of the magnetic field) and inverse aspect ratio $\epsilon = a/R_0$ much lower than unity, MHD equations can be simplified by expanding in powers of $\epsilon$. This procedure has been widely used, since the first paper of Strauss [98], both for analytical and numerical work. At the leading order in $\epsilon$, $O(\epsilon^2)$, and considering the low-\(\beta\) approximation, $\beta \approx O(\epsilon^2)$, the reduced-MHD equations describe the plasma in the cylindrical approximation. Toroidal corrections enter the equations at the next order in the inverse aspect ratio expansion. The derivation of these equations has been described in detail in ref. [99] and is only briefly reported here.

Following the low-\(\beta\) tokamak ordering, it is possible to write

$$\frac{v_\perp}{v_A} \approx \frac{B_\perp}{B_\varphi} \approx \frac{\mathbf{B} \cdot \nabla}{\nabla} \approx O(\epsilon),$$

$$\frac{v_\varphi}{v_A} \approx \frac{\nabla \cdot \mathbf{v}_\perp}{v_A / a} \approx \frac{\nabla (RB_\varphi)}{B_\varphi} \approx O(\epsilon^2), \quad \frac{\partial}{\partial t} \approx \frac{v_A}{R}.$$ Here, a cylindrical-coordinate system $(R, Z, \varphi)$ has been used, and the subscript $\perp$ denotes components perpendicular to $\nabla \varphi$. The magnetic field can be written as

$$\mathbf{B} = \left( F_0 + \hat{E} \right) \nabla \varphi + R_0 \nabla \psi \times \nabla \varphi + O(\epsilon^3 B_\varphi),$$

where $\psi$ is the poloidal magnetic flux function, $F_0 = R_0 B_0$, $B_0$ is the vacuum (toroidal) magnetic field at $R = R_0$, and $F \approx O(\epsilon^2 F_0)$ is given, at the leading order, by equilibrium corrections. Substituting eq. (177) and Ohm’s law, eq. (10), into Faraday’s law, eq. (5), we obtain

$$R_0 \frac{\partial \psi}{\partial t} \nabla \varphi + c \left( \eta \mathbf{J} - \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) = -\epsilon \nabla \phi + O(\epsilon^4 v_A B_\varphi),$$

where $\phi$ is the scalar potential. Taking the cross product by $\nabla \varphi$, eq. (178) can be solved with respect to $v_\perp$:

$$v_\perp = -\frac{cR^2}{R_0 B_0} \nabla \psi \times \nabla \varphi + O(\epsilon^3 v_A).$$

Equation (179) states that, at the lowest order, the perpendicular velocity is given by the $\mathbf{E} \times \mathbf{B}$ drift. Then, taking the $\nabla \varphi$ component of eq. (178), the following equation for the evolution of the poloidal magnetic flux function is obtained:

$$\frac{\partial \psi}{\partial t} = \frac{cR^2}{R_0 B_0} \nabla \psi \times \nabla \varphi \cdot \nabla \phi - \frac{c}{R_0} \frac{\partial \phi}{\partial \varphi} + \eta \frac{c^2}{4\pi} \Delta^* \psi + O(\epsilon^4 v_A B_\varphi),$$

with the Grad-Shafranov operator $\Delta^*$ defined by eq. (170).

Upon applying the operator $\nabla \varphi \cdot \nabla \times R^2 \ldots$ to the momentum equation, eq. (2), the following equation for the evolution of the scalar potential is obtained:

$$\frac{d}{d \tau} \left( D \frac{\partial \phi}{\partial \tau} - \frac{2c}{R_0 B_0} \frac{\partial \phi}{\partial Z} \right) \nabla^2 \psi + \nabla \phi \cdot \left( D \frac{\partial \phi}{\partial \tau} - \frac{c}{R_0 B_0} \frac{\partial \phi}{\partial Z} \right) \nabla \phi =$$

$$-\frac{B_0}{4\pi c} \mathbf{B} \cdot \nabla \Delta^* \psi - \frac{B_0}{c R_0} \nabla \cdot \left( R^2 \nabla P \times \nabla \varphi \right) + O(\epsilon^4 v_A^2 B_\varphi),$$

(181)
where

\[ \dot{\phi} = \frac{R^2}{R_0^3} \phi, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_\perp \cdot \nabla, \]

\[ \nabla^2 \perp = \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{R}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} \right). \]

Note that, both in eq. (180) and eq. (181), \( v_\perp \) and \( \vec{F} \) enter only at the fourth order in \( \epsilon \). In eq. (181) the dependence on the density gradient has been retained explicitly. With the particular choice of the mass density \( \rho R^3 = \dot{\rho} R_0^3 = \text{const} \), and using the definition of \( \mathbf{v}_\perp \) given in eq. (179), the continuity equation, eq. (1), is satisfied up to the third order. The pressure equation becomes

\[ \frac{DP}{Dt} = O(\epsilon^4 v_A B^2 \frac{a}{\epsilon^1}). \]

5'4. Hybrid MHD-kinetic models. In order to include in this model the effects on an energetic-ion population, we can take advantage from the fact that the energetic particle density is typically much smaller than the bulk plasma density. The following ordering can then be adopted:

\[ \frac{n_H}{n_i} \approx O(\epsilon^3), \quad \frac{T_H}{T_i} \approx O(\epsilon^{-2}), \]

where \( n_H \) (\( n_i \)) and \( T_H \) (\( T_i \)) are the energetic particle (bulk ion) density and temperature respectively. Thus, the following ordering for the ratio of the energetic to bulk ion beta follows:

\[ \frac{\beta_H}{\beta_i} \approx O(\epsilon), \]

It can be shown [100] that, making use of the above ordering, the MHD momentum equation is modified by a term which represent the perpendicular component of the divergence of the energetic-particle stress tensor \( \Pi_H \) (in ref. [100] an alternative, equivalent form in which the electric current associated to the energetic ions appears, instead of the energetic-particle stress tensor, is also derived). Thus, the \( O(\epsilon^3) \) equation for the evolution of the scalar potential is modified as follows

\[ \dot{\phi} \left( \frac{D}{Dt} - \frac{2c}{R_0 B_0} \frac{\partial \phi}{\partial Z} \right) \nabla^2 \phi + \nabla \dot{\phi} \cdot \left( \frac{D}{Dt} - \frac{c}{R_0 B_0} \frac{\partial \phi}{\partial Z} \right) \nabla \phi = 

\[ -\frac{B_0}{4\pi c} \mathbf{B} \cdot \nabla \Delta^* \psi - \frac{B_0}{4\pi c} \mathbf{B} \cdot \nabla \cdot \left[ R^2 (\nabla P + \nabla \cdot \Pi_H) \times \nabla \varphi \right] + O(\epsilon^4 \frac{v_A^4 B^2}{a^2 c}). \]

In order to close the set of reduced MHD eqs. (180) and (183), the hot-particle stress-tensor components can be evaluated by directly calculating the appropriate velocity moment of the distribution function for the particle population moving in the perturbed fields \( \psi \) and \( \phi \) (see appendix sect. 5'5).

5'5. Hybrid MHD-kinetic code. In this section, we describe the code that solves the \( O(\epsilon^3) \) reduced MHD model, in the limit of zero bulk-plasma pressure. In such limit, only eqs. (180) and (183) need to be solved. As a boundary condition, we take a rigid conducting wall at the plasma edge. The numerical tool [61, 72, 73] used to solve the \( O(\epsilon^3) \) model is based on a field solver originating from an existing \( O(\epsilon^2) \) code [101]. Such field solver uses toroidal coordinates \( (r, \vartheta, \varphi) \), finite differences in the radial direction \( (r) \) and Fourier expansion in the poloidal \( (\vartheta) \) and toroidal \( (\varphi) \) directions. The coupled
equations for the Fourier components of the magnetic and velocity stream functions $\psi$ and $\phi$ are advanced in time using a semi-implicit algorithm, where all the linear terms that couple with the cylindrical part of the equilibrium (i.e., the component having poloidal and toroidal mode numbers $(m, n) = (0, 0)$) are treated implicitly. The non-linear terms, the terms which arise from the toroidal corrections to the cylindrical approximation and the contributions of the energetic particles (the term containing $\nabla \cdot \mathbf{H}$ in eq. (183)) are treated explicitly. Moreover, only the Fourier components in a half plane of the $(m, n)$ space are evolved, the ones that fall in the other half plane being recovered from the reality condition of the solution:

\[
\hat{\psi}_{m,-n}(r, t) = \hat{\psi}_{m,n}(r, t), \quad \hat{\phi}_{m,-n}(r, t) = \hat{\phi}_{m,n}(r, t).
\]

The equilibrium configuration used for numerical simulations can be exactly calculated to the desired order in $\epsilon$, starting from the expression for the equilibrium toroidal current

\[
\Delta^* \psi^{eq} = -\frac{4\pi}{c} \frac{R}{R_0} J_{0\varphi},
\]

and expanding $\psi^{eq}$ in powers of $\epsilon$,

\[
\psi^{eq}(r, \vartheta) = \psi^{eq}_0(r) + \psi^{eq}_1(r, \vartheta) + O(\epsilon^2 \psi^{eq}_0).
\]

In the toroidal coordinate system $(r, \vartheta, \varphi)$ the Grad-Shafranov operator can be expressed as

\[
\Delta^* = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2} - \frac{1}{R} \left( \cos \vartheta \frac{\partial}{\partial r} - \sin \vartheta \frac{\partial}{\partial \vartheta} \right),
\]

with $R = R_0 + r \cos \vartheta$. To the leading order, that is in the cylindrical approximation, eq. (185) is given by

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d \psi^{eq}_0}{dr} \right) = -\frac{4\pi}{c} \frac{R}{R_0} J_{0\varphi}[\psi^{eq}_0(r)],
\]

yielding

\[
\frac{d \psi^{eq}_0}{dr} = \frac{B_0}{R_0} q(r),
\]

which can be integrated assigning $\psi^{eq}_0(a) = 0$ and $q(r)$, the safety factor in the cylindrical ($O(\epsilon^2)$) approximation ($q(r) = rB_0/(RB_0)$). Equation (186) gives the symmetric $(m = 0, n = 0)$ Fourier component of the poloidal magnetic flux function $\psi^{eq}_0$. To the next order in $\epsilon$, eq. (185) yields

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi^{eq}_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi^{eq}_1}{\partial \vartheta^2} - \frac{1}{R_0 \cos \vartheta} \frac{d \psi^{eq}_0}{dr} = -\frac{4\pi}{c} \frac{B_0}{R_0} \frac{d}{d \psi^{eq}_0} \left( \frac{R}{R_0} J_{0\varphi} \right)_{\psi^{eq} = \psi^{eq}_0}.
\]

Equation (187) admits solutions of the form

\[
\psi^{eq}_1(r, \vartheta) = \psi^{eq}_1(r) \cos \vartheta = \Delta(r) \cos \vartheta \frac{d \psi^{eq}_0}{dr},
\]

where we have introduced the so-called Shafranov shift $\Delta$. Substituting eq. (188) into eq. (187) and using the leading order solution of eq. (186), the following equation for the
Shafranov shift $\Delta$ is obtained:

$$
\frac{1}{r} \frac{d}{dr} \left[ r \left( \frac{d\psi^q}{dr} \right)^2 \frac{d\Delta}{dr} \right] - \frac{1}{R_0} \left( \frac{d\psi^q}{dr} \right)^2 = 0 .
$$

Equation (189) can be integrated assigning the radial derivative of the Shafranov shift at the center $\Delta'(0) = 0$ (regularity condition) and $\Delta(a) = 0$ (corresponding to $\psi^q = 0$ on the rigid conducting wall), to obtain $\Delta(r)$. The substitution of $\Delta(r)$ into eq. (188) allows us to obtain the first-order $(1, 0)$ Fourier component of the magnetic flux function $\psi^q$, thus completing the equilibrium solution at the desired order. Note that, once fixed $r$ and $\vartheta$, the quantity $\Delta(r)$ corresponds to the shift, with respect to the center of the poloidal cross section, of the geometric center of the magnetic surface labelled by the value $\psi^q(r, \vartheta)$. Such a shift causes shear Alfvén waves, even when propagating along the magnetic field line, to cross the radial grid, thus imposing restrictions on the time step of integration [99]. Further restrictions are imposed by the strength of the explicitly solved terms (as, e.g., in the case of high inverse aspect ratio equilibria and/or highly non-linear cases).

The term $\Pi_H$ in eq. (183) is the pressure-tensor of the energetic (hot) ions; it can be expressed in terms of the corresponding distribution function $f_H (\Pi_H \equiv m_H \int d^3v \Pi_H$, with $m_H$ being the energetic-ion mass), to be determined by solving the Vlasov equation (the collisionless limit of the Boltzmann equation). Since the time scale of the dynamics we want to analyze is long compared to a cyclotron period, it is convenient [102, 103] to solve the Vlasov equation in the gyrocenter-coordinate system $\mathbf{x} \equiv (\mathbf{R}, \mathbf{M}, \mathbf{U}, \vartheta)$, where $\mathbf{R}$ is the gyrocenter position, $\mathbf{M}$ is the magnetic moment, $\mathbf{U}$ is the parallel velocity (i.e., the velocity along the magnetic-field line), and $\vartheta$ is the gyrophase. This corresponds to averaging the single-particle equations of motion over many cyclotron orbits and allows one to retain the relevant finite-Larmor-radius effects without resolving the details of the gyromotion. Such a choice is particularly suited for numerical time integration of the particle motion, as the numerical-stability constraint on the time-step size turns out to be much less severe than that we would obtain without adopting the averaging procedure.

The hot-particle pressure tensor assumes the following form, in terms of the gyrocenter-coordinate distribution function,

$$
\Pi_H (t, x) = \frac{1}{m_H} \int d^3u D_{z \rightarrow \mathbf{Z}} F_H (t, \mathbf{R}, \mathbf{M}, \mathbf{U}) \times \left[ \frac{\Omega_H M}{m_H} I + \mathbf{b} \left( \mathbf{U} - \frac{\Omega_H M}{m_H} \right) \right] \delta (x - \mathbf{R}) ,
$$

where $I$ is the identity tensor ($I_{ij} \equiv \delta_{ij}$), $F_H (t, \mathbf{R}, \mathbf{M}, \mathbf{U})$ is the energetic-particle distribution function in gyrocenter coordinates, and $D_{z \rightarrow \mathbf{Z}}$ is the Jacobian of the transformation from canonical to gyrocenter coordinates.

The Vlasov equation can be written as

$$
\frac{dF_H}{dt} = 0 ,
$$

with

$$
\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dZ}{dt} \frac{\partial}{\partial Z} .
$$
and \( d\mathbf{Z} / dt \) given by the following equations of motion [61, 104, 105, 106]

\[
\frac{d\mathbf{R}}{dt} = \mathbf{U} \mathbf{b} + \frac{e_H}{m_H \Omega_H} \mathbf{b} \times \nabla \phi - \frac{\mathbf{U}}{m_H \Omega_H} \mathbf{b} \times \nabla a || + \\
\left( \frac{\mathbf{\Omega} + a ||}{m_H \Omega_H} \left( \mathbf{U} + \frac{a ||}{m_H} \right) \right) \mathbf{b} \times \nabla \ln B,
\]

\( \frac{d\mathbf{M}}{dt} = 0, \)

\[
\frac{d\mathbf{U}}{dt} = \frac{1}{m_H} \dot{\mathbf{b}} \cdot \left\{ \frac{e_H}{\Omega_H} \left[ \mathbf{U} + \frac{a ||}{m_H} \right] \nabla \phi + \frac{\mathbf{\Omega} + a ||}{m_H \Omega_H} \nabla a || \right\} \times \nabla \ln B + \\
- \frac{\Omega_H \mathbf{M}}{m_H} \mathbf{b} \cdot \nabla \ln B.
\]

Here, \( e_H \) and \( \Omega_H \equiv e_H B / (m_H c) \) are, respectively, the energetic-ion charge and Larmor frequency. The fluctuating potential \( a || \) is related to the poloidal magnetic flux function \( \psi \) through the relationship

\( a || = \frac{e_H \mathbf{R}_0}{c \mathbf{R}} \psi. \)

Note that the magnetic moment \( \mathbf{M} \) is exactly conserved in this coordinate system and that, correspondingly, neither \( \mathbf{F}_H \) nor the equations of motion contain any dependence on the gyrophase \( \theta \).

The particle-simulation approach to the solution of Vlasov equation, eq. (191), consists in representing any phase-space function \( G(t, \mathbf{Z}) \) by its discretized form,

\[
G(t, \mathbf{Z}) = \sum_{l=1}^{N_{\text{part}}} \Delta_l(t) G(t, \mathbf{Z}_l(t)) \delta(\mathbf{Z} - \mathbf{Z}_l(t)),
\]

where \( \Delta_l \) is the volume element around the phase-space marker \( \mathbf{Z}_l \), and in assuming that each marker evolves in time according to the gyrocenter equations of motion, eqs. (192). Such markers can then be interpreted as the phase-space coordinates of a set of \( N_{\text{part}} \) “particles”, and \( G(t, \mathbf{Z}) \) can be approximated by

\[
G(t, \mathbf{Z}) \approx \sum_{l=1}^{N_{\text{part}}} \Delta_l(t) G(t, \mathbf{Z}_l(t)) \delta(\mathbf{Z} - \mathbf{Z}_l(t)).
\]

The time-variation of the volume element \( \Delta_l(t) \) is then given by

\[
\frac{d\Delta_l}{dt} = \Delta_l(t) \left( \frac{\partial}{\partial \mathbf{Z}} \frac{d\mathbf{Z}}{dt} \right)_{t, \mathbf{Z}_l(t)}.
\]

For the purpose of calculating the pressure tensor components, eq.(190), it is convenient to directly represent the quantity \( D_{z e - \mathbf{Z}} \mathbf{F}_H \) according to its discretized form

\[
D_{z e - \mathbf{Z}}(t, \mathbf{Z}) \mathbf{F}_H (t, \mathbf{Z}) \approx \sum_{l=1}^{N_{\text{part}}} \overline{m}_l(t) \delta(\mathbf{Z} - \mathbf{Z}_l(t)),
\]

with the weight factor \( \overline{m}_l \) defined by

\[
\overline{m}_l(t) \equiv \Delta_l \mathbf{F}_H (t, \mathbf{Z}_l(t)),
\]
and
\begin{equation}
\Sigma_t \equiv \Delta_t (t) \ D_{z_e \to z} \left( t, Z_t (t) \right).
\end{equation}
In fact, from eqs. (191), (196), and from the Liouville theorem,
\begin{equation}
\frac{\partial}{\partial t} D_{z_e \to z} + \frac{\partial}{\partial Z} \left( D_{z_e \to z} \frac{dZ}{dt} \right) = 0,
\end{equation}
it is immediate to show that
\begin{equation}
\frac{d\Sigma_t}{dt} = 0,
\end{equation}
and
\begin{equation}
\frac{d\Pi_t}{dt} = 0.
\end{equation}

At each time step, the fluctuating electromagnetic potentials are computed at the grid points of a three-dimensional toroidal domain in terms of the Fourier components yielded by the field solver. Phase-space coordinates are then evolved in the fluctuating fields, and the pressure tensor components at the grid points are updated, in order to close the MHD equations for the next time step.

Field values at each particle position are obtained by trilinear interpolation of the fields at the vertices of the cell the particle belongs to. The corresponding trilinear weight function is adopted, after pushing the particles, in order to distribute their contribution to the pressure tensor components among the vertices of the cell. Phase-space coordinates and weights for the simulation particles are initially determined in such a way to yield a prescribed (e.g., Maxwellian) distribution function. Particle pushing is performed by integrating eqs. (192) by a second-order Runge-Kutta method, more accurate than the standard \( O(\Delta t) \) Euler method (\( O(\Delta t^2) \) is properly retained), although more time consuming. Particles that hit the wall \( (r = a) \) are considered lost and are not re-injected in the plasma.

It has been shown [61, 107, 108, 109, 110, 111] that, as far as regimes are considered where the distribution function can be expected to slightly depart from the equilibrium one, it is worth limiting the numerical investigation to the evolution of the perturbed part \( \delta F_H \), defined by the relationship
\begin{equation}
F_H (t, \overline{R}, \overline{M}, \overline{U}) = F_{H0} (t, \overline{R}, \overline{M}, \overline{U}) + \delta F_H (t, \overline{R}, \overline{M}, \overline{U}),
\end{equation}
where \( F_{H0} \) is the lowest-order ("equilibrium") distribution function.

In terms of \( \delta F_H \), eq. (191) can be written in the form
\begin{equation}
\frac{d\delta F_H}{dt} = S,
\end{equation}
with
\[ S \equiv - \frac{dF_{H0}}{dt}. \]

Meanwhile, eq. (197) is replaced by the following one,
\begin{equation}
D_{z_e \to z} (t, Z) \delta F_H (t, Z) \approx \sum_{i=1}^{N_{\text{part}}} \overline{w}_i (t) \delta \left( Z - Z_t (t) \right),
\end{equation}
with
\[ w_l(t) \equiv \Delta_l \delta F_H(t, Z_l(t)) , \]
and
\[ \frac{d w_l}{dt} = \Delta_l S(t, Z_l(t)) . \]

Note that eq. (203) is by no means equivalent to a linearization of the Vlasov equation, since all non-linear terms are correctly retained. The decomposition of eq. (203) is useful in reducing numerical noise as long as \( |\delta F_H| \ll |F_H| \).

In the present paper, when adopting the \( \delta F \) approach, we take \( F_{H0} \) to be Maxwellian
\[ F_{H0} \propto n_H(\mathbf{R}) \exp \left( -\frac{\Omega_H \mathbf{M} + \frac{1}{2} m_H U^2}{T_H} \right) , \]
where \( n_H(\mathbf{R}) \) and \( T_H \) are, respectively, the energetic-particle equilibrium density and (uniform) temperature. The r.h.s. of eq. (204) is then given by
\[ S(t, \mathbf{R}, \mathbf{M}, U) = -F_{H0} \left\{ \frac{d \mathbf{R}}{dt} \nabla \ln n_H + \frac{e_H}{T_H} \left[ \frac{M}{m_H} + \frac{U}{\Omega_H} \left( U + \frac{a_0}{m_H} \right) \right] \mathbf{b} \times \nabla \ln B \cdot \nabla \phi + \right. \\
\left. \frac{e_H U}{T_H \Omega_H m_H} \mathbf{b} \cdot \nabla \phi \times \nabla a_0 \right\} . \]

We also assume the following model for the energetic-particle equilibrium density:
\[ n_H(r) = n_{H0} \exp \left( -\left( \frac{r^2}{L_n^2} \right)^{\alpha_n} \right) , \]
where \( n_{H0} \) is the on-axis density.

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